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DECAY AND ANALYTICITY OF SOLITARY WAVES

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ABSTRACT. – Considered here are detailed aspects of solitary-wave solutions of nonlinear evolution equations including the Euler equations for the propagation of gravity waves on the surface of an ideal, incompressible, inviscid fluid. Two properties will occupy our attention. The first, described already in an earlier paper, concerns the regularity of these travelling waves. In the context of certain classes of model equations for long waves in nonlinear dispersive media, we showed that solitary waves are obtained as the restriction to the real axis of functions analytic in a strip of the form $\{z : -a < \Im(z) < a\}$ in the complex plane. In this direction, the scope of our previous discussion of model equations is broadened considerably. Moreover, it is also shown that solitary-wave solutions of the full Euler equations have the properties that the free surface is given by the restriction to the real axis of a function analytic in a strip in the complex plane and the velocity potential is the restriction to the flow domain of a function that is analytic in an open set in complex 2-space $\mathbb{C}^2$. The second issue considered is the asymptotic decay of solitary waves to a quiescent state away from their principal elevation. A theorem pertaining to the evanescence of solutions of certain types of one-dimensional convolution equations is formulated and proved, showing that decay is related to the smoothness of the Fourier transform of the convolution kernel k, as well as the nonlinearity present in the equation. It is demonstrated that if the Fourier transform $\hat{k} \in H^s$ for some $s > 1/2$, the rate of decay of a solution is at least as fast as that of the kernel k itself. This result is used to establish asymptotic properties of solitary-wave solutions of a broad class of model equations, and of solitary-wave solutions of the full Euler equations.

RÉSUMÉ. – Sont traités, dans cet article, quelques aspects détaillés des ondes solitaires solutions d'équations d'évolutions non linéaires, incluant les équations d'Euler pour la propagation des ondes gravitationnelles à la surface d'un fluide idéal, incompressible et non visqueux. Deux propriétés ont attiré notre attention. La première, déjà décrite dans un article antérieur concerne la régularité de ces ondes de translations. Dans le cadre de certaines classes d'équations modélisant les ondes longues dans un milieu non linéaire et dispersif, nous avons montré que les ondes solitaires s'obtiennent comme la restriction à l'axe réel de fonctions analytiques dans une bande de la forme $\{z : -a < \Im(z) < a\}$ du plan complexe. Dans cette perspective, l'étendue de notre précédente discussion sur les équations modèles est considérablement élargie. En outre, il est aussi montré que les ondes solitaires solutions des équations d'Euler complètes ont les propriétés que la surface libre est donnée par la restriction à l'axe réel d'une fonction analytique dans une bande du plan complexe et que le potentiel des vitesses est la restriction au domaine du fluide d'une fonction analytique dans un ouvert d'un espace à deux dimensions sur $\mathbb{C}^2$. La seconde propriété considérée est la décroissance asymptotique des ondes solitaires vers un état au repos éloigné de leur maximum principal. Un théorème concernant l'évanescence des solutions de certains types d'équations de convolution unidimensionnelles est énoncé et prouvé, montrant que la décroissance est liée à la régularité de la transformée de Fourier du noyau k de convolution ainsi qu'à la non linéarité de l'équation. Il est démontré que si la transformée de Fourier $\hat{k}$ appartient à H^s pour $s > 1/2$, le taux de décroissance de la solution est au moins aussi rapide que celui du noyau. Le résultat est utilisé pour établir des propriétés asymptotiques des ondes solitaires solutions d'une large classe d'équations modèles, et en particulier pour les ondes solitaires solutions des équations d'Euler complètes.

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1. Introduction

This paper is concerned with two aspects of solitary waves that are a reflection of their natural appearance as smooth, steadily propagating disturbances of elevation or depression, asymptotically approaching a constant level on either side of their crests. These attributes of the real phenomenon find mathematical expression in regularity theory and decay results for solitary-wave solutions of nonlinear wave equations. It is our purpose here to investigate these mathematical properties in the context of model equations of Korteweg-de Vries-type, regularized long-wave-type and nonlinear Schrödinger-type, as well as for the time-dependent Euler equations for the propagation of gravity waves on the surface of an inviscid, incompressible fluid.

This program was begun in our earlier paper [14] where we considered model equations of Korteweg-de Vries-type (KdV-type)

$$u_t + u_x + u^p u_x - (Mu)_x = 0, \tag{1.1}$$

regularized long-wave-type (RLW-type)

$$u_t + u_x + u^p u_x + (Mu)_t = 0, \tag{1.2}$$

and Schrödinger-type

$$iu_t - Mu + |u|^p u = 0. \tag{1.3}$$

Here, p is a positive integer, $u = u(x,t)$ is a function of the two real variables x and t and subscripts connote partial differentiation. The dependent variable u often represents a displacement or a velocity in physical contexts, x is usually related to the spatial variable in the primary direction of propagation, while t is typically proportional to elapsed time. The operator M which results from modelling dispersion is a Fourier multiplier operator defined by

$$\widehat{Mv}(\xi) = \alpha(\xi)\hat{v}(\xi), \tag{1.4}$$

where a circumflex surmounting a function of the spatial variable denotes that function's Fourier transform and the symbol α is measurable and even, so that M maps real-valued functions to other real-valued functions. A solitary-wave solution of (1.1) or (1.2) is a travelling-wave solution $\varphi(x - ct)$ of the evolution equation, where c is a positive constant and φ is an even function, usually but not always of one sign, and tending to zero at $\pm\infty$. Solitary-wave solutions of (1.3) have the form $e^{i\omega t}\phi(x - \theta t)$ where ϕ has the same general form as outlined for φ. The existence of solitary-wave solutions for (1.1), (1.2) and (1.3) corresponding to a broad range of symbols, and including more general nonlinearities than these appearing above, has been studied by Albert *et al.* [1], Benjamin *et al.* [5] and Weinstein [22]. It is demonstrated in these papers that functions φ or ϕ representing solitary waves are infinitely differentiable and, along with all their derivatives, members of L_2 and L_1. In [14], it was shown on the basis of quite reasonable assumptions on α

that such solitary waves φ or ϕ are in fact the restriction to the real axis of other functions Φ, say, that are holomorphic in a strip of the form

$$\{z : -\sigma_0 < \Im(z) < \sigma_0\} \tag{1.5}$$

in the complex plane $\mathbb{C}$, where $\sigma_0 > 0$ depends on the solitary wave in question.

In the present study, model equations of the form (1.1)-(1.3) will be considered in which the nonlinearity is considerably generalized to those having the form

$$F(u)_x = F'(u)u_x \tag{1.1-1.2a}$$

in (1.1) and (1.2) and the form

$$F(|u|)u \tag{1.3a}$$

in (1.3), where $F : \mathbb{R} \to \mathbb{R}$ is a suitably smooth real-valued function of a real variable. Complementing the analyticity results to be derived in this setting, it will be shown that the regularity of the function $\hat{k}(\xi) = 1/(1 + \alpha(\xi))$ determines how fast the solitary-wave solutions of these model equations decay at infinity. Precisely, if $\hat{k} \in H^s$ for some $s > 1/2$, then the corresponding solitary-wave solution decays to zero at $\pm\infty$ at least as rapidly as does $1/|x|^s$. Moreover, if $\hat{k}$ has an analytic extension to a strip of the form displayed in (1.5) for some $\sigma_0 > 0$ and satisfies the condition

$$\sup_{|\eta| \le \sigma} \int_{-\infty}^{\infty} |\hat{k}(\xi + i\eta)|^2 d\xi < \infty \tag{1.6}$$

for any σ with $0 < \sigma < \sigma_0$, then the corresponding solitary-wave solution decays exponentially at the same order as does k.

The discussion centered on the model equations (1.1)-(1.3) illustrates in bold relief the importance played by dispersion. Both the result of analyticity and that of decay of solitary waves depend strongly on the presence of dispersion in the form of a symbol α that grows appropriately at $\pm\infty$. Of course, the nonlinearity is also important, especially for the existence of such travelling waves.

Perhaps the best known of the model equations (1.1)–(1.3) is the Korteweg-de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.7}$$

itself. Its solitary-wave solutions $\varphi(x - ct)$, $c > 1$, have the exact form

$$\varphi_c(z) = 3\mathrm{C}\sec^2\left(\frac{\sqrt{\mathrm{C}}}{2}z\right), \tag{1.8}$$

where $\mathrm{C} = c - 1$ and $z = x - ct$. They exemplify the properties of analyticity and decay under discussion here. Indeed, the function φ_c in (1.8) extends to a function analytic in the strip (1.5) with $\sigma_0 = \pi/\sqrt{\mathrm{C}}$ and φ_c decays exponentially with the asymptotic form $3Ce^{-\sqrt{C}|x|}$ as $|x| \to \infty$. These aspects of φ_c, which are obvious from its explicit form, will be seen to follow from our general theory. In this case $\alpha(\xi) = \xi^2$, so that $\hat{k}(\xi) = (C + \xi^2)^{-1}$. Thus $\hat{k}$ extends analytically to a strip of the form in (1.5), $\hat{k}$ satisfies (1.6) there and $k(x) = \frac{1}{2\sqrt{C}} e^{-\sqrt{C}|x|}$ decays exponentially.

The KdV equation (1.7) has been derived as a model for quite a number of physical situations, but it arose first as a physical model in [6] and in [11] where it was put forward as a model derived from the Euler equations for the propagation of two-dimensional small-amplitude, long-wavelength, water waves in a channel. In the present context, it is natural to inquire whether or not the properties of analyticity and decay that are apparent for the solitary-wave solutions (1.8) of the KdV equation (1.7) are shared with the solitary-wave solutions of the full, two-dimensional Euler equations.

Extensive studies have been carried out of solitary-wave solutions of the full Euler equations for two-dimensional motions on the surface of an incompressible, inviscid and irrotational flow in a horizontal channel in the absence of surface tension effects. To form the problem, let Ω be the flow region and let the coordinate axes be chosen to move at the constant speed $c > 0$ with the wave so that the y-axis passes through the crest and the flow is steady in this frame of reference. Also, let $\phi(x, y)$ be the velocity potential and $\psi(x, y)$ the stream function associated with the flow, both defined on the closure of the flow domain Ω. The flow is normalized so that the horizontal bottom $y = 0$ is the stream line $\psi(x, y) = 0$, the free surface is the stream line $\psi(x, y) = 1$, and the solitary-wave profile is $y = H(x)$ for $x \in \mathbb{R}$, where $H(x)$ is an even function, monotone decreasing on $\mathbb{R}^+$. Thus $\Omega = \{(x, y) : x \in \mathbb{R}, 0 < y < H(x)\}$. The equations for the problem as just formulated are as follows:

$$
(1.9) \qquad \begin{cases} \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy = \frac{1}{2}c^2 + gh & \text{for} \quad y = H(x), \\ \phi_x H_x - \phi_y = 0 & \text{for} \quad y = H(x), \\ \Delta\phi = 0 & \text{in} \quad \Omega, \\ \phi_y = 0 & \text{for} \quad y = 0, \end{cases}
$$

where h is the depth of the undisturbed fluid so that $H(x) \to h$ as $x \to \pm\infty$ and g is the gravity constant. Conventionally, the consideration of equations (1.9) begins with a change of independent variables from x and y to ϕ and ψ. The advantage of viewing $x = x(\phi, \psi)$ and $y = y(\phi, \psi)$ is that they are formally harmonic functions defined on the known strip $D = \{(\phi, \psi) : -\infty < \phi < \infty, 0 < \psi < 1\}$ and continuous up to the horizontal boundaries of D. Thus questions of existence and so on are reduced to problems defined on a fixed domain.

The existence theory of solitary waves for the problem (1.9) has been developed over a period of more than four decades. In 1947, Lavrentiev [12] proved the existence of small-amplitude solitary waves as the limit of periodic wavetrains. Friedrichs and Hyers [8] proved the existence of solitary waves under the condition that the Froude number F, defined by $F^2 = c^2/gh$, is greater than, but close to one. Noticing the relation between the existence of solitary waves and the value of the Froude number, Keady and Pritchard [10] proved that solitary waves which are symmetric about their crest and strictly decreasing away from the crest are only possible when $F > 1$ and $H(x) \geq h$ for all x.

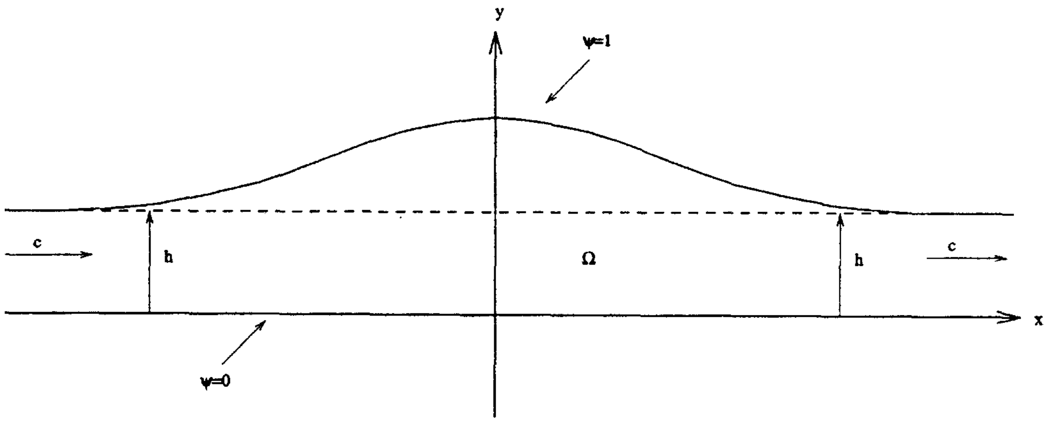

Fig. 1. The solitary wave.

McLeod [17] derived the same result and showed that the total mass of the fluid is finite, *i.e.* $\int_{-\infty}^{\infty}(H(x)-h)dx<\infty$. Amick and Toland [3] completed the proof of the existence of solitary waves for any Froude number F in the range $(1, F_c)$, where F_c is the Froude number associated with the so-called wave of greatest height. They also showed that the solitary-wave profile $H(x)$ is real-analytic. In another paper [2], they demonstrated that (1.9) has a family of periodic solutions which converge to a solitary-wave solution. Afterwards, Craig and Sternberg [7] resolved the question of whether or not (1.9) has solitary-wave solutions which are not symmetric or not monotone. They proved by using the method of moving planes that when the Froude number $F>1$, any solitary-wave solution of (1.9) has an elevation which is symmetric and monotone on either side of its crest with $H(x)>h$ for all x. Thus, a necessary and sufficient condition for there to be a solitary wave on the surface of an ideal fluid in a channel has been obtained, namely that the Froude number F lies in the range $(1, F_c)$, or what is the same, its wave speed c is greater than the linear, nondispersive, long-wave speed $\sqrt{gh}$ and bounded above appropriately. Later, Benjamin, Bona and Bose [5] used topological methods to verify the existence of solitary waves for $F\in(1,F_c)$. In their paper, the following equation was derived:

$$\omega(\phi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k(\phi-t)\frac{\gamma\sin\omega(t)}{1+3\gamma\int_0^t\sin\omega(\tau)d\tau}dt, \tag{1.10}$$

where $\theta=-\omega$ is the angle between the free surface of the solitary wave and the x-axis, and the constant γ and the function k are defined by

$$\gamma=\frac{g}{u_0^3}\quad\text{and}\quad k(\phi)=\sqrt{\frac{2}{\pi}}\ln\left(\coth\frac{\pi|\phi|}{4}\right),$$

respectively, where u_0 is the speed of the solitary wave at its crest. They showed that there is a solution of (1.10) which is an odd function on the real line and nonnegative on $(0,\infty)$.

Our contribution to the theory of solitary-wave solutions of the two-dimensional Euler equations is to show that such solutions are the restriction of functions analytic in a strip in one or two complex dimensions, depending on which dependent variable is being considered. To accomplish this, we will rely upon the integral equation (1.10). The exponential decay of the solitary wave, discussed already in [3] and [7], will also follow as a corollary of our general approach.

It is worth noting some of the numerical approximations of solitary-wave solutions of the Euler equations (Longuet-Higgins and Fenton [15], [16], and Hunter and Vanden-Broeck [9]) that have informed our collective view, especially of large-amplitude waves.

The outline of this paper is as follows. After introducing some notation in Section 2, a theory of decay properties of solitary-wave solutions of model equations is presented in Section 3. In Section 4, interest is concentrated on analyticity of solitary-wave solutions of model equations of the form (1.1)-(1.3) with general forms of nonlinearity and of the full Euler equations. The text finishes with a short conclusion and an appendix containing a proof of a technical point.

2. Notation

For any complex number $z \in \mathbb{C}$, the real part and the imaginary part of z are denoted by $\Re z$ and $\Im z$, respectively.

By $L_p = L_p(\mathbb{R})$ for p in the range $1 \leq p \leq \infty$, we mean the standard class of pth-power Lebesgue-integrable functions on the real line $\mathbb{R}$ with the usual modification if $p = \infty$. The standard norm of a function $f \in L_p$ will be denoted by $\| f \|_p$. The inner product of two functions f and g in L_2, denoted by (f, g), is the integral

$$(f, g) = \int_{\infty}^{\infty} f(x)\overline{g(x)}dx,$$

where the overbar connotes complex conjugation.

The space of all infinitely differentiable functions with compact support in $\mathbb{R}$ is denoted by $C_c^\infty = C_c^\infty(\mathbb{R})$.

The Fourier transform of a measurable function ϕ defined on $\mathbb{R}$ is written $\hat{\phi}$ and is defined to be

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x)e^{i\xi x}dx.$$

The inverse Fourier transform of ϕ, denoted by $\check{\phi}$, is defined as

$$\check{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\zeta)e^{-ix\zeta}d\zeta.$$

For any $s \in \mathbb{R}$, the Sobolev space $H^s = H^s(\mathbb{R})$ consists of all functions f which are tempered distributions such that $\|f\|_{(s)} = \left(\int_{-\infty}^{\infty}(1 + |\zeta|^2)^s|\hat{f}(\zeta)|^2 d\zeta\right)^{\frac{1}{2}} < \infty$. The space H^∞ is the intersection $H^\infty = \bigcap_{s\in\mathbb{R}} H^s$.

The convolution of two functions f and g defined on $\mathbb{R}$ is written $f * g$, and $f \circ g$ is the frequently appearing, related integral

$$(f \circ g)(x) = \int_{-\infty}^{\infty} \overline{f(t-x)}\, g(t)\, dt.$$

By a *solitary-wave solution* of (1.1) or (1.2) for a given, general nonlinearity $F(u)_x$ and a dispersion symbol α, we shall mean a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that φ is a continuous function with $\lim\limits_{|x| \to \infty} \varphi(x) = 0$, and such that for some positive constant $c > 1$, $\varphi(x - ct)$ defines a weak solution of (1.1) or (1.2) in the sense that

$$(\varphi, ((c-1)I + M)g') - (F(\varphi), g') = 0$$

or

$$(\varphi, ((c-1)I + cM)g') - (F(\varphi), g') = 0$$

for any $g \in C_c^{\infty}(\mathbb{R})$, where the inner product is that of L_2. A similar definition will be adopted later for solutions of (1.3). As already mentioned, the existence of such solutions for a wide range of symbols α and certain classes of nonlinearities F has been dealt with in the recent works of Albert *et al.* [1], Amick and Toland [4], Benjamin *et al.* [5], and Weinstein [22].

For any $x \in \mathbb{R}$, the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$.

3. Decay properties of solitary waves

In this section, we begin with a study of solutions to a class of nonlinear convolution equations of the form

$$f = k * G(f), \tag{3.0.1}$$

where f is an unknown function, k is a given integral kernel satisfying certain decay conditions to be specified presently, and $G = G(u)$ is a measurable function, bounded on bounded sets, and satisfying the growth condition $|G(u)| \le M|u|^r$ for all small values of $|u|$ and some constants $M > 0$ and $r > 1$. The results obtained in the discussion of (3.0.1) will be applied to solitary-wave solutions of the model equations (1.1), (1.2) and (1.3) with general nonlinearity F, as well as to those of the full Euler equations.

3.1. Decay of solutions to the convolution equation (3.0.1)

The following preliminary lemma states inequalities which will find use in establishing the ensuing main theorem.

LEMMA 3.1.1. – *Let l and m be constants satisfying $0 < l < m - 1$. Then there is a constant $B > 0$ depending only on l and m such that the inequalities*

$$I = \int_0^{\infty} \frac{t^l}{(1+\epsilon t)^m (1+|x-t|)^m}\, dt \le \frac{B|x|^l}{(1+\epsilon|x|)^m},$$

$$II = \int_{-\infty}^{0} \frac{|t|^l}{(1+\epsilon|t|)^m (1+|x-t|)^m}\, dt \le \frac{B|x|^l}{(1+\epsilon|x|)^m}$$

hold for any ϵ with $0 < \epsilon \leq 1$ and any real x with $|x| \geq 1$, while the inequalities

$$III = \int_0^\infty \frac{dt}{(1+\epsilon t)^m (1+|x-t|)^m} \leq \frac{B}{(1+\epsilon|x|)^m},$$
$$IV = \int_{-\infty}^0 \frac{dt}{(1+\epsilon|t|)^m (1+|x-t|)^m} \leq \frac{B}{(1+\epsilon|x|)^m}$$

are valid for the same range of ϵ and any $x \in \mathbb{R}$.

Proof. – First, consider the case when $x \geq 1$. Write the integral I as

$$I = \int_0^x \frac{t^l}{(1+\epsilon t)^m (1+x-t)^m}\, dt + \int_x^\infty \frac{t^l}{(1+\epsilon t)^m (1+t-x)^m}\, dt = I_1 + I_2.$$

Since $\frac{1}{(1+\epsilon t)(1+x-t)} = \frac{1}{1+\epsilon+\epsilon x}\left[\frac{\epsilon}{1+\epsilon t} + \frac{1}{1+x-t}\right]$,

$$I_1 \leq \frac{2^{m-1} x^l}{(1+\epsilon+\epsilon x)^m} \int_0^x \left[\frac{\epsilon^m}{(1+\epsilon t)^m} + \frac{1}{(1+x-t)^m}\right] dt \leq \frac{2^m x^l}{(m-1)(1+\epsilon x)^m},$$

whereas, I_2 can be estimated straightforwardly by

$$\begin{aligned} I_2 &\leq \frac{1}{(1+\epsilon x)^m} \int_0^\infty \frac{(y+x)^l\, dy}{(1+y)^m} \\ &\leq \frac{x^l}{(1+\epsilon x)^m} \int_0^\infty \frac{dy}{(1+y)^{m-l}} = \frac{x^l}{(m-l-1)(1+\epsilon x)^m}. \end{aligned}$$

It follows from these estimates of I_1 and I_2 that there is a constant B independent of both ϵ and x in the ranges being considered such that

$$I \leq \frac{B|x|^l}{(1+\epsilon|x|)^m}. \tag{3.1.1}$$

When $x \leq -1$, $|t+x| \leq t - x$ for any $t > 0$. Therefore, it transpires that

$$I \leq \int_0^\infty \frac{t^l}{(1+\epsilon t)^m (1+|t+x|)^m}\, dt.$$

Since $-x \geq 1$, it follows from the above argument that (3.1.1) still holds for $x \leq -1$ and $0 < \epsilon \leq 1$. The inequalities for II, III and IV may be obtained in a similar way. □

THEOREM 3.1.2. – *Suppose that $f \in L_\infty$ with $\lim_{|x|\to\infty} f(x) = 0$ is a solution of the convolution equation*

$$f(x) = \int_{-\infty}^\infty k(x-t) G(f(t))\, dt,$$

where the measurable function G satisfies $|G(u)| \leq |u|^r$ for any $u \in \mathbb{R}$ and some constant $r > 1$ and the kernel k is also a measurable function satisfying the condition $\hat{k} \in H^s$ for *some* $s > \frac{1}{2}$. *Then* $f \in L_1 \cap L_2$ *and there exists a constant* l *with* $0 < l < s$ *such that* $|x|^l f(x) \in L_2 \cap L_\infty$.

Proof. – If $s < \infty$, choose an $l > 0$ such that $s > l + \frac{1}{2}$, otherwise choose any constant $l > 0$ and another number, still denoted by s for the sake of convenience, with $s > l + \frac{1}{2}$. It follows from Lemma 3.1.1 that the inequality

$$\int_0^\infty \frac{t^{2l}\,dt}{(1+\epsilon t)^{2s}(1+|x-t|)^{2s}} \le \frac{B|x|^{2l}}{(1+\epsilon|x|)^{2s}} \tag{3.1.2}$$

holds for some constant B and any x and ϵ with $|x| \ge 1$ and $0 < \epsilon \le 1$, where B is independent of both these variables.

Define h_ϵ by $h_\epsilon(x) = \frac{|x|^l}{(1+\epsilon|x|)^s} f(x)$ for $x \in \mathbb{R}$. Since $f \in L_\infty$, $h_\epsilon(x) \in L_2$, and because $r > 1$ and $\lim\limits_{|x|\to\infty} f(x) = 0$, for any $\delta > 0$, there is a constant $N \ge 1$ such that

$$|f(x)|^{r-1} \le \delta, \tag{3.1.3}$$

for almost all $|x| > N$. Choose $\delta < \frac{1}{2^{s+1}B^{1/2}\|\hat{k}\|_{(s)}}$ and let N be such that (3.1.3) holds for the chosen value of δ. Then using the Schwarz inequality leads to the following estimate:

$$\begin{aligned}
\int_N^\infty |h_\epsilon(x)|^2 dx &\le \int_N^\infty |h_\epsilon(x)| \frac{|x|^l}{(1+\epsilon|x|)^s} \int_{-\infty}^\infty |k(x-t)G(f(t))|\,dt\,dx \\
&\le \int_N^\infty |h_\epsilon(x)| \frac{|x|^l}{(1+\epsilon|x|)^s} \left(\int_{-\infty}^\infty (1+|x-t|)^{2s}|k(x-t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{-\infty}^\infty \frac{|G(f(t))|^2 dt}{(1+|x-t|)^{2s}}\right)^{\frac{1}{2}} dx \\
&\le 2^s\|\hat{k}\|_{(s)} \left(\int_N^\infty |h_\epsilon(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_N^\infty \frac{x^{2l}}{(1+\epsilon x)^{2s}} \int_{-\infty}^\infty \frac{|G(f(t))|^2 dt}{(1+|x-t|)^{2s}} dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.1.4}$$

Applying Fubini's theorem and the inequalities (3.1.2) and (3.1.3) to (3.1.4) yields

$$\begin{aligned}
&\left(\int_N^\infty |h_\epsilon(x)|^2 dx\right)^{\frac{1}{2}} \\
&\le 2^s\|\hat{k}\|_{(s)} \left(\int_{-\infty}^\infty |G(f(t))|^2 \int_N^\infty \frac{x^{2l}\,dx}{(1+\epsilon x)^{2s}(1+|t-x|)^{2s}}\,dt\right)^{\frac{1}{2}} \\
&\le 2^s B^{\frac{1}{2}}\|\hat{k}\|_{(s)} \left(\left(\int_{-\infty}^{-N} + \int_N^\infty\right) |G(f(t))|^2 \frac{|t|^{2l}}{(1+\epsilon|t|)^{2s}} dt\right)^{\frac{1}{2}} \\
&\quad + 2^s\|\hat{k}\|_{(s)} \left(\int_{-N}^N |G(f(t))|^2 \int_N^\infty \frac{x^{2l}\,dx}{(1+\epsilon x)^{2s}(1+|t-x|)^{2s}}\,dt\right)^{\frac{1}{2}} \\
&\le 2^s\delta B^{\frac{1}{2}}\|\hat{k}\|_{(s)} \left(\left(\int_{-\infty}^{-N} + \int_N^\infty\right) |f(t)|^2 \frac{|t|^{2l}}{(1+\epsilon|t|)^{2s}}\,dt\right)^{\frac{1}{2}}
\end{aligned} \tag{3.1.5}$$

$$+ 2^s\|\hat{k}\|_{(s)}\left(\int_{-N}^{N}|G(f(t))|^2dt\int_{-\infty}^{\infty}\frac{(|x|+N)^{2l}\,dx}{(1+|x|)^{2s}}\right)^{\frac{1}{2}}$$

$$= 2^s\delta B^{\frac{1}{2}}\|\hat{k}\|_{(s)}\left(\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)|h_\epsilon(t)|^2dt\right)^{\frac{1}{2}}$$

$$+ 2^s M\|\hat{k}\|_{(s)}\left(\int_{-N}^{N}|G(f(t))|^2dt\right)^{\frac{1}{2}},$$

where $M = \left(\int_{-\infty}^{\infty}\frac{(|x|+N)^{2l}\,dx}{(1+|x|)^{2s}}\right)^{\frac{1}{2}}$. A similar computation yields the inequality

$$\left(\int_{-\infty}^{-N}|h_\epsilon(x)|^2dx\right)^{\frac{1}{2}} \leq 2^s\delta B^{\frac{1}{2}}\|\hat{k}\|_{(s)}\left(\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)|h_\epsilon(t)|^2dt\right)^{\frac{1}{2}}$$
$$+ 2^s M\|\hat{k}\|_{(s)}\left(\int_{-N}^{N}|G(f(t))|^2dt\right)^{\frac{1}{2}}. \tag{3.1.6}$$

Adding inequalities (3.1.5) and (3.1.6) leads to the estimate

$$\left(\int_{-\infty}^{-N}|h_\epsilon(x)|^2dx\right)^{\frac{1}{2}} + \left(\int_{N}^{\infty}|h_\epsilon(x)|^2dx\right)^{\frac{1}{2}}$$
$$\leq 2^{s+1}\delta B^{\frac{1}{2}}\|\hat{k}\|_{(s)}\left(\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)|h_\epsilon(t)|^2dt\right)^{\frac{1}{2}}$$
$$+ 2^{s+1}M\|\hat{k}\|_{(s)}\left(\int_{-N}^{N}|G(f(t))|^2dt\right)^{\frac{1}{2}},$$

or, because of the restriction on δ,

$$\left(\int_{-\infty}^{-N}|h_\epsilon(x)|^2dx\right)^{\frac{1}{2}} + \left(\int_{N}^{\infty}|h_\epsilon(x)|^2dx\right)^{\frac{1}{2}}$$
$$\leq \frac{2^{s+1}M\|\hat{k}\|_{(s)}}{1-2^{s+1}\delta B^{\frac{1}{2}}\|\hat{k}\|_{(s)}}\left(\int_{-N}^{N}|G(f(t))|^2dt\right)^{\frac{1}{2}}. \tag{3.1.7}$$

Let $\epsilon \to 0$ and apply Fatou's Lemma to (3.1.7) to obtain

$$\left(\int_{-\infty}^{-N}|x|^{2l}|f(x)|^2dx\right)^{\frac{1}{2}} + \left(\int_{N}^{\infty}x^{2l}|f(x)|^2dx\right)^{\frac{1}{2}}$$
$$\leq \frac{2^{s+1}M\|\hat{k}\|_{(s)}}{1-2^{s+1}\delta B^{\frac{1}{2}}\|\hat{k}\|_{(s)}}\left(\int_{-N}^{N}|G(f(t))|^2dt\right)^{\frac{1}{2}},$$

which implies $(1+|x|^l)f(x) \in L_2$ or, what is the same, $\hat{f} \in H^l$. Since $f(x)$ and $1/(1+\epsilon|x|)^s$ both lie in L_2, it follows that for any $\epsilon > 0$, $f(x)/(1+\epsilon|x|)^s \in L_1$. To show that $f \in L_1$, one may estimate the following integral by using Fubini's theorem, the Schwarz inequality, Lemma 3.1.1 and Inequality (3.1.2):

$$(3.1.8)\quad \begin{aligned} \int_N^\infty \frac{|f(x)|\,dx}{(1+\epsilon x)^s} &\le \int_{-\infty}^\infty |G(f(t))| \int_N^\infty \frac{|k(x-t)|\,dx}{(1+\epsilon x)^s}\,dt \\ &\le \int_{-\infty}^\infty |G(f(t))| \left(\int_N^\infty (1+|x-t|)^{2s}|k(x-t)|^2 dx\right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_N^\infty \frac{dx}{(1+|x-t|)^{2s}(1+\epsilon x)^{2s}}\right)^{\frac{1}{2}} dt \\ &\le 2^s \|\hat{k}\|_{(s)} B^{\frac{1}{2}} \int_{-\infty}^\infty \frac{|G(f(t))|\,dt}{(1+\epsilon|t|)^s} \\ &\le 2^s \|\hat{k}\|_{(s)} \delta B^{\frac{1}{2}} \left(\int_{-\infty}^{-N} + \int_N^\infty\right) \frac{|f(t)|\,dt}{(1+\epsilon|t|)^s} + 2^s \|\hat{k}\|_{(s)} B^{\frac{1}{2}} \int_{-N}^N |G(f(t))|\,dt. \end{aligned}$$

Arguing similarly leads to

$$(3.1.9)\quad \begin{aligned} \int_{-\infty}^{-N} \frac{|f(x)|\,dx}{(1+\epsilon|x|)^s} &\le 2^s \|\hat{k}\|_{(s)} \delta B^{\frac{1}{2}} \left(\int_{-\infty}^{-N} + \int_N^\infty\right) \frac{|f(t)|\,dt}{(1+\epsilon|t|)^s} \\ &\quad + 2^s \|\hat{k}\|_{(s)} B^{\frac{1}{2}} \int_{-N}^N |G(f(t))|\,dt. \end{aligned}$$

It follows from (3.1.8) and (3.1.9) that

$$\left(\int_{-\infty}^{-N} + \int_N^\infty\right) \frac{|f(t)|\,dt}{(1+\epsilon|t|)^s} \le \frac{2^{s+1}\|\hat{k}\|_{(s)} B^{\frac{1}{2}}}{1-2^{s+1}\|\hat{k}\|_{(s)}\delta B^{\frac{1}{2}}} \int_{-N}^N |G(f(t))|\,dt.$$

Using Fatou's lemma as $\epsilon \to 0$ gives

$$\left(\int_{-\infty}^{-N} + \int_N^\infty\right) |f(t)|\,dt \le \frac{2^{s+1}\|\hat{k}\|_{(s)} B^{\frac{1}{2}}}{1-2^{s+1}\|\hat{k}\|_{(s)}\delta B^{\frac{1}{2}}} \int_{-N}^N |G(f(t))|\,dt,$$

and hence $f \in L_1$.

To show $|x|^l f(x) \in L_\infty$, multiply both sides of the equation $f = k * G(f)$ by $|x|^l$ and apply the inequality $|x|^l \le \beta(|x-t|^l + |t|^l)$ with $\beta = \max\{1, 2^{l-1}\}$ to obtain the estimate

$$(3.1.10)\quad |x|^l |f(x)| \le \beta \int_{-\infty}^\infty |x-t|^l |k(x-t) G(f(t))|\,dt + \beta \int_{-\infty}^\infty |k(x-t)||t|^l |G(f(t))|\,dt.$$

Since k, $|x|^l k$, $G(f)$ and $|x|^l G(f)$ all lie in L_2, it follows that the right-hand side of (3.1.10) is the convolution of L_2-functions, and therefore is a bounded function. □

Remark. – It is clear that Theorem 3.1.2 still holds if $|G(u)|$ is bounded on any bounded subset of $\mathbb{R}$ and the inequality $|G(u)| \le M|u|^r$, where $M > 0$ and $r > 1$, is valid for sufficiently small values of u. These conditions combined with the fact that $f \in L_\infty$ imply that $(1+|x|)^l G(f(x)) \in L_2$ if and only if $|x|^l G(f(x))$ is square integrable on $(-\infty, -N] \cup [N, \infty)$ for some constant $N > 0$. The just mentioned conditions on the function G are also sufficient to obtain the other results in this section. For the sake of simplicity and without real loss of generality, we shall continue to assume $|G(u)| \le |u|^r$ to hold for any $u \in \mathbb{R}$ throughout Section 3.

Interest is now focussed on finding the largest number $l > 0$ such that $|x|^l f(x) \in L_2$. Corollary 3.1.3, Corollary 3.1.4, Theorem 3.1.5 and Theorem 3.1.6 are concerned with this issue. The outcome of our analysis is that $l \ge s$, which is to say that f decays to zero at infinity at least at the same rate as does k.

COROLLARY 3.1.3. – Under the conditions of Theorem 3.1.2, $\hat{f} \in H^s$, and $|x|^\nu f(x) \in L_\infty$, where ν is any constant with $0 < \nu \le s$ if $s < \infty$ and ν is any positive number if $s = \infty$.

Proof. – Let l and s be defined as in the proof of Theorem 3.1.2, and let $\nu_1 = \min\{s, rl\}$. Then, replace l by ν_1 in (3.1.10) and consider the resulting inequality. Since $|x|^{\nu_1} k(x) \in L_2$ and $G(f) \in L_1$, it follows from Young's inequality that the first integral on the right-hand side of the modified version of (3.1.10) is an L_2-function. Because $|G(f(t))||t|^{\nu_1} \le (|f(t)||t|^{\frac{\nu_1}{r}})(|f(t)||t|^{\frac{\nu_1}{r}})^{r-1}$ and $f(t)|t|^{\frac{\nu_1}{r}} \in L_2 \cap L_\infty$, it is deduced that $G(f(t))|t|^{\nu_1} \in L_2$. When combined with Young's inequality and the fact that $k \in L_1$, the latter point implies that the second integral on the right-hand side of (3.1.10) is also an L_2-function. Hence, it is seen that $f(x)|x|^{\nu_1} \in L_2$. If $\nu_1 = rl < s$, one may use the above argument to show that $f(x)|x|^{\nu_2} \in L_2$ for $\nu_2 = \min\{s, r^2 l\}$. Then repeating this argument at most finitely many times leads to the conclusion $\hat{f} \in H^s$. If $\hat{k} \in H^\infty$, then $l > 0$ can be chosen as any positive number, and thus $\hat{f} \in H^\infty$. □

COROLLARY 3.1.4. – *Suppose that the functions f and G satisfy the conditions in Theorem 3.1.2 and that there is a constant $\sigma_0 > 0$ such that the integral kernel k satisfies the inequality*

$$\int_{-\infty}^{\infty} e^{2\sigma|x|}|k(x)|^2\, dx < \infty,$$

for any σ with $0 \le \sigma < \sigma_0$. Then $e^{\sigma|x|}f(x) \in L_1 \cap L_\infty$ for any $\sigma \in [0, \sigma_0)$.

Proof. – Since $\hat{k} \in H^\infty$, it follows from Corollary 3.1.3 that $|x|^s f(x) \in L_1 \cap L_\infty$ for any $s > 0$. We shall first obtain bounds for $\|(\cdot)^n f(\cdot)\|_1$ for any integer $n \ge 0$, where $(\cdot)$ connotes the function $h(x) = x$, and then use them to show that f decays exponentially.

Fix a constant c with $0 < c < \sigma_0$ and let

$$M_1 = \max\{\|(\cdot)|f(\cdot)|^{r-1}\|_\infty, \|G(f)\|_1\},$$

$$M_2 = \left(\int_{-\infty}^{\infty} |k(x)|^2 e^{2c|x|}\, dx\right)^{\frac{1}{2}}, \quad \text{and}$$

$$K = \max\left\{\frac{c}{2}\|f\|_1, \sqrt{\frac{c^2}{3!}\|(\cdot)f(\cdot)\|_1}, \sqrt[3]{\frac{c^3}{4!}\|(\cdot)^2 f(\cdot)\|_1}, 4, 4M_1M_2\sqrt{c}\right\}.$$

Clearly we have

$$\|(\cdot)^l f(\cdot)\|_1 \le \frac{(l+2)!K^{l+1}}{c^{l+1}} \tag{3.1.11}$$

for $l = 0, 1, 2$. Induction is now used to show that (3.1.11) holds for all integers $l \ge 0$. Suppose that there is an integer $n \ge 2$ such that (3.1.11) is valid for any integer l with $0 \le l \le n$. When $l = n+1$, one may estimate the L_1-norm of $x^{n+1}f(x)$ by Young's inequality as follows:

$$\begin{aligned} &\|(\cdot)^{n+1} f(\cdot)\|_1 \\ &= \|(\cdot)^{n+1}(k * G(f(\cdot)))\|_1 \\ &\le \sum_{j=0}^{n+1} \binom{n+1}{j} \||(\cdot)^{n+1-j}k(\cdot)| * |(\cdot)^j G(f(\cdot))|\|_1 \\ &\le \sum_{j=0}^{n+1} \binom{n+1}{j} \|(\cdot)^{n+1-j}k(\cdot)\|_1 \|(\cdot)^j G(f(\cdot))\|\|_1. \end{aligned} \tag{3.1.12}$$

Applying the Schwarz inequality and the definition of M_2 to $\|(\cdot)^{n+1-j}k(\cdot)\|_1$ yields

$$\begin{aligned} &\|(\cdot)^{n+1-j}k(\cdot)\|_1 \\ &\le \left(\int_{-\infty}^{\infty} |k(x)|^2 e^{2c|x|}\, dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |x|^{2(n+1-j)} e^{-2c|x|}\, dx\right)^{\frac{1}{2}} \\ &\le \sqrt{2} M_2 \sqrt{\frac{(2n+2-2j)!}{(2c)^{2(n+1-j)+1}}} \le 2M_2 \frac{(n+1-j)!}{c^{n+1-j+1/2}}. \end{aligned} \tag{3.1.13}$$

For any j with $1 \le j \le n+1$, estimate the other terms on the right-hand side of (3.1.12) using the quantity M_1 defined above:

$$\begin{aligned} &\|(\cdot)^j G(f(\cdot))\|\|_1 \\ &= \int_{-\infty}^{\infty} |x|^j |G(f(x))|\, dx \\ &\le \int_{-\infty}^{\infty} |x|^{j-1} |f(x)| (|x|^{\frac{1}{r-1}} |f(x)|)^{r-1}\, dx \\ &\le M_1 \int_{-\infty}^{\infty} |x|^{j-1} |f(x)|\, dx = M_1 \|(\cdot)^{j-1} f(\cdot)\|_1. \end{aligned} \tag{3.1.14}$$

Now applying (3.1.13), (3.1.14), the induction hypothesis and the definition of K to (3.1.12) gives

$$\begin{aligned}
&\|(\cdot)^{n+1}f(\cdot)\|_1 \\
&\le 2M_1M_2\frac{(n+1)!}{c^{n+1+1/2}} + 2M_1M_2\sum_{j=1}^{n+1}\binom{n+1}{j}\frac{(n+1-j)!}{c^{n+1-j+1/2}}\|(\cdot)^{j-1}f(\cdot)\|_1 \\
&\le 2M_1M_2\left(\frac{(n+1)!}{c^{n+1+1/2}} + \sum_{j=1}^{n+1}\binom{n+1}{j}\frac{(n+1-j)!}{c^{n+1-j+1/2}}\frac{(j+1)!K^j}{c^j}\right) \\
&= \frac{2M_1M_2(n+1)!}{c^{n+1+1/2}}\left(1+\frac{(n+2)K^{n+3}-(n+3)K^{n+2}-K^2+2K}{(K-1)^2}\right) \\
&\le \frac{(n+3)!K^{n+2}}{c^{n+2}}\left(\frac{2}{(n+2)(n+3)}+\frac{1}{n+2}\right) \le \frac{(n+3)!K^{n+2}}{c^{n+2}}.
\end{aligned}$$

This completes the inductive step and it is thereby concluded that Inequality (3.1.11) holds for all integers $l \ge 1$.

Applying (3.1.11) shows that

$$\begin{aligned}
&\int_{-\infty}^{\infty}|f(x)|e^{\nu|x|}dx \\
&\le \sum_{l=0}^{\infty}\frac{\nu^l}{l!}\int_{-\infty}^{\infty}|x|^l|f(x)|\,dx \\
&\le \sum_{l=0}^{\infty}\frac{\nu^l}{l!}\frac{(l+2)!\,K^{l+1}}{c^{l+1}} = \sum_{l=0}^{\infty}(l+1)(l+2)\nu^lK^{l+1}/c^{l+1}.
\end{aligned}$$

Hence, if $0<\nu<c/K$, then $\int_{-\infty}^{\infty}|f(x)|e^{\nu|x|}dx<\infty$.

To finish the proof, let $\nu_0=\sup\left\{\nu>0:\int_{-\infty}^{\infty}|f(x)|e^{\nu|x|}dx<\infty\right\}$ and observe the following inequality:

$$\begin{aligned}
|f(x)|e^{\nu|x|} &\le \int_{-\infty}^{\infty}|k(x-t)|e^{\nu|x-t|}|G(f(t))|e^{\nu|t|}dt \\
&\le \int_{-\infty}^{\infty}|k(x-t)|e^{\nu|x-t|}|f(t)|e^{\nu|t|}|f(t)|^{r-1}dt.
\end{aligned} \tag{3.1.15}$$

If $0<\nu<\min\{\nu_0,\sigma_0\}$, then $f(x)e^{\nu|x|}\in L_2$, which results from (3.1.15) and the facts $f(x)e^{\nu|x|}\in L_1$, $f(x)\in L_\infty$ and $k(x)e^{\nu|x|}\in L_2$. In consequence, we have $f(x)e^{\nu|x|}\in L_\infty$. Assume $\nu_0<\sigma_0$ and choose a constant $\nu>0$ such that $\frac{\nu_0}{r}<\nu<\min\{\nu_0,\frac{\sigma_0}{r}\}$. It follows that $k(x)e^{r\nu|x|}\in L_1$ and $f(x)e^{\nu|x|}\in L_1\cap L_\infty$. Replacing ν by $r\nu$ in (3.1.15), the right-hand side of the inequality is seen to be the convolution of the L_1-functions $k(x)e^{r\nu|x|}$ and $\left(|f(x)|e^{\nu|x|}\right)^r$, viz.

$$|f(x)|e^{r\nu|x|} \le \int_{-\infty}^{\infty}|k(x-t)|e^{r\nu|x-t|}|f(t)|e^{\nu|t|}\left(|f(t)|e^{\nu|t|}\right)^{r-1}dt,$$

and thus $f(x)e^{r\nu x} \in L_1 \cap L_2$ which is contrary to the definition of ν_0. Hence, $\nu_0 \geq \sigma_0$, which is to say that f decays at least at the same order as does k. □

In the next two theorems, sufficient conditions are formulated implying a solution f of the equation $f = k * G(f)$ decays to zero at infinity at the same order as k.

THEOREM 3.1.5. – *Suppose that $f = k*G(f)$, where f, k and G satisfy the assumptions in Theorem 3.1.2. Suppose also there is a constant $m > 1$ such that $\lim_{|x|\to\pm\infty} |x|^m k(x) = C_\pm$, where C_+, $C_- \in \mathbb{C}$ and the adornments $\pm$ correspond to limits at $+\infty$ and $-\infty$, respectively. Then it follows that*

$$\lim_{x\to\pm\infty} |x|^m f(x) = C_\pm \int_{-\infty}^{\infty} G(f(t))\,dt.$$

Proof. – The identity $\lim_{|x|\to\infty} x^m f(x) = C_+ \int_{-\infty}^{\infty} G(f(t))\,dt$ is verified. The limit at $-\infty$ may be proved in the same way.

First, it is shown that for any l with $0 < l < m$, $|x|^l f(x) \to 0$ as $|x| \to \infty$, and then this fact is used to verify the advertised conclusion. Choose any $l_1 > 0$ such that $m - 1/2 < l_1 < \min\{m, r(m-1/2)\}$. It follows from the hypothesis on k and Corollary 3.1.3 that $|x|^{l_1} k(x) \to 0$ and $|x|^{l_1}|G(f(x))| \leq \left(|f(t)||x|^{l_1/r}\right)^r \to 0$ as $|x| \to \infty$. Moreover, we know that k, $G(f) \in L_1 \cap L_2$. Applying these facts to the inequality

$$|x|^{l_1}|f(x)| \leq \beta \int_{-\infty}^{\infty} |x-t|^{l_1}|k(x-t)G(f(t))|\,dt + \beta \int_{-\infty}^{\infty} |k(x-t)||t|^{l_1}|G(f(t))|\,dt$$

shows that the right-hand side goes to 0 as $|x| \to \infty$, and thus $|x|^{l_1} f(x) \to 0$, as $|x| \to \infty$. Note that we can take $\beta = \min\{1, 2^{l_1 - 1}\}$. If $l_1 < m$, then choose an l_2 with $l_1 < l_2 < \min\{m, l_1 r\}$ and repeat the above argument to get $|x|^{l_2} f(x) \to 0$, as $|x| \to \infty$. Continuing this argument a finite number of times leads to the desired conclusion.

Since $\lim_{|x|\to\infty} |x|^m k(x) = C_\pm$, there are constants $N_0 > 0$ and $A > 0$ such that $|x|^m |k(x)| \leq A$ and $|k(x)| \leq A$ when $|x| \geq N_0$. For any $\epsilon > 0$, there is an $N \geq N_0$ such that

$$\int_N^\infty |G(f(x))|\,dx < \epsilon, \quad \int_N^\infty |G(f(x))|^2\,dx < \epsilon^2, \quad \int_{-\infty}^{-N} |G(f(x))|\,dx < \epsilon,$$

$$\int_N^\infty |k(x)|\,dx < \epsilon, \quad \int_{-\infty}^{-N} |k(x)|\,dx < \epsilon, \quad \text{and} \quad |t|^m |G(f(t))| < \epsilon$$

for any $|t| > N$. Additionally, it may be assumed that $|x^m k(x-t) - C_+| < \epsilon$ for any $t \in (-N, N)$ and any $x \geq N + N_0$. Then the estimate

$$\begin{aligned}
&\left| f(x)x^m - C_+ \int_{-\infty}^{\infty} G(f(t))\,dt \right| \\
&\leq \left| \int_{-N}^{N} (x^m k(x-t) - C_+) G(f(t))\,dt \right| + \left| C_+ \left(\int_{-\infty}^{-N} + \int_N^\infty \right) G(f(t))\,dt \right|
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{m-1}\left[\int_{-\infty}^{-N} |x-t|^m |k(x-t)G(f(t))|\,dt + \int_{-\infty}^{-N} |k(x-t)G(f(t))||t|^m\,dt\right] \\
&+ 2^{m-1}\left(\int_{N}^{x-N_0} + \int_{x-N_0}^{x+N_0} + \int_{x+N_0}^{\infty}\right)|x-t|^m|k(x-t)G(f(t))|\,dt \\
&+ 2^{m-1}\int_{N}^{\infty} |k(x-t)G(f(t))||t|^m\,dt \\
&\le [\|G(f)\|_1 + 2|C_+| + 2^m 3A + 2^{m-1}N_0^m\|k\|_2 + 2^m\|k\|_1]\epsilon
\end{aligned}
$$

holds when x is sufficiently large. Since ϵ is arbitrary, it follows that

$$
f(x) \sim \frac{C_+}{x^m}\int_{-\infty}^{\infty} G(f(t))\,dt, \qquad \text{as } x \to +\infty. \qquad \square
$$

Note. – In Theorem 3.1.5, the condition $m > 1$ is needed for the existence of a constant $s > 1/2$ such that $\hat{k} \in H^s$, so that Theorem 3.1.2 applies to the discussion.

In the next theorem, we shall consider the case wherein k decays exponentially.

THEOREM 3.1.6. – *Suppose $f = k * G(f)$, where k and G satisfy the hypotheses in Theorem 3.1.2. Suppose also that for some $\sigma_0 > 0$,*

$$
\lim_{x\to\pm\infty} e^{\sigma_0|x|}k(x) = C_\pm.
$$

Then the function f satisfies the relations $\sup_{x\in\mathbb{R}} e^{\sigma_0|x|}|f(x)| < \infty$ and

$$
\lim_{x\to\pm\infty} e^{\sigma_0|x|}f(x) = C_\pm \int_{-\infty}^{\infty} e^{\pm\sigma_0 t}G(f(t))\,dt,
$$

where, as before, C_+, C_- and $\pm$ correspond to the limits at $+\infty$ and $-\infty$, respectively.

Proof. – It follows from Corollary 3.1.4 that $f(x)e^{\frac{\sigma_0}{r}|x|} \in L_1 \cap L_\infty$. The inequality

$$
|G(f(x))|e^{\sigma_0|x|} \le \left(|f(x)|e^{\frac{\sigma_0}{r}|x|}\right)^r
$$

then implies that $|G(f(x))|e^{\sigma_0|x|} \in L_1 \cap L_\infty$. The remainder of the proof follows the argument in the proof of the last theorem. $\square$

Remark. – It follows from Theorem 3.1.2 that the integral kernel k also has a smoothing effect on its solutions. Any solution f satisfying $f \in L_\infty$ and $\lim_{|x|\to\infty} f(x) = 0$ must be a continuous and bounded function on $\mathbb{R}$, even though $k(x)$ itself may not be bounded and continuous. This point will come to the fore for the Benjamin-Ono equation and the full Euler equations. (The proposition becomes obvious upon noting f is the convolution of the L_2-functions $k(x)$ and $G(f(x))$.)

One question not considered in this paper is the case when the integral kernel k of Equation (3.0.1) decays to zero at infinity like $1/|x|^m$ for some real number m with

$0 < m \leq 1$. Under this assumption, the argument used in this paper may not be effective, since $\hat{k}$ is not in H^s for some $s > 1/2$ and k is not in L_1. One example in hand indicates that the solution of Equation (3.0.1) may not decay as fast as the kernel k. We expect to discuss this issue at a later stage using L_p-based Sobolev spaces for values of p other than 2.

We now turn to the application of these results about convolution equations of the form displayed in (3.0.1) to the solitary waves that are our primary focus. As will become apparent momentarily, the solitary-wave solutions of the model equations in (1.1), (1.2) or (1.3) may be realized as solutions of convolution equations as in (3.0.1). The same is also true of solitary-wave solutions of the Euler equations as seen already in (1.10).

3.2. Decay of solitary-wave solutions

We begin the discussion with the KdV-type equations in (1.1). Assume that for some $c > 1$, $u(x,t) = \varphi(x - ct)$ is a solitary-wave solution of the equation

$$u_t + u_x + F(u)_x - Mu_x = 0.$$

Under conditions to be stated precisely in Theorem 3.2.1, one concludes that φ is a solution of the convolution equation $\varphi = \frac{1}{\sqrt{2\pi}} k_c * F(\varphi)$, where the Fourier transform $\hat{k}_c$ of k_c is given explicitly as $\hat{k}_c(\xi) = 1/(c - 1 + \alpha(\xi))$ and $\alpha(\xi) \geq 0$ is the symbol of the dispersion operator M. We shall show next that for any $c > 1$, $\hat{k}_c \in H^s$ if and only if $\hat{k} = \hat{k}_2 = 1/(1 + \alpha) \in H^s$ for some $s > 1/2$. Therefore, without loss of generality, it will only be required to discuss the case $c = 2$ in the proof of Theorem 3.2.1.

Suppose that $\hat{k}(\xi) = 1/(1 + \alpha(\xi)) \in H^s$ for some $s > 1/2$. It then follows from standard Sobolev-embedding results and an elementary application of the Cauchy-Schwarz inequality that $k \in L_1$, $\hat{k} \in C_b(\mathbb{R})$ and $\hat{k}(\xi) \to 0$ as $|\xi| \to \infty$. Of course, $\hat{k} \in L_2$ as well. Moreover, for any $c > 1$, the quantity $\frac{1+\alpha(\xi)}{c-1+\alpha(\xi)}$ is bounded above by $B = \max\{1, \frac{1}{c-1}\}$ and hence the same properties just stated for $\hat{k}$ accrue to $\hat{k}_c$. For any $t \in \mathbb{R}$, it follows from Plancherel's theorem that

$$\|\hat{k}_c(\cdot + t) - \hat{k}_c(\cdot)\|_2^2 = \int_{-\infty}^{\infty} \left| \frac{1}{c - 1 + \alpha(\xi + t)} - \frac{1}{c - 1 + \alpha(\xi)} \right|^2 d\xi$$
$$= \int_{-\infty}^{\infty} |k_c(x)|^2 |e^{itx} - 1|^2 dx = \int_{-\infty}^{\infty} |k_c(x)|^2 |2 \sin \frac{tx}{2}|^2 dx.$$

On the other hand, a different calculation shows that

$$\|\hat{k}_c(\cdot + t) - \hat{k}_c(\cdot)\|_2^2$$
$$= \int_{-\infty}^{\infty} \frac{(1 + \alpha(\xi + t))^2 (1 + \alpha(\xi))^2}{(c - 1 + \alpha(\xi + t))^2 (c - 1 + \alpha(\xi))^2} \left| \frac{1}{1 + \alpha(\xi + t)} - \frac{1}{1 + \alpha(\xi)} \right|^2 d\xi$$
$$\leq B^4 \int_{-\infty}^{\infty} \left| \frac{1}{1 + \alpha(\xi + t)} - \frac{1}{1 + \alpha(\xi)} \right|^2 d\xi = B^4 \int_{-\infty}^{\infty} |k(x)|^2 |2 \sin \frac{tx}{2}|^2 dx.$$

In consequence, we see that

$$\text{(3.2.1)} \quad \int_{-\infty}^{\infty} |k_c(x)|^2 \left| 2 \sin \frac{tx}{2} \right|^2 dx = \|\hat{k}_c(\cdot + t) - \hat{k}_c(\cdot)\|_2^2 \leq B^4 \int_{-\infty}^{\infty} |k(x)|^2 \left| 2 \sin \frac{tx}{2} \right|^2 dx.$$

If $s < 1$, multiply both sides of (3.2.1) by $1/|t|^{1+2s}$, integrate the resulting inequality with respect to t over the real line $\mathbb{R}$ and use Fubini's theorem to obtain

$$\int_{-\infty}^{\infty} |k_c(x)|^2 dx \int_{-\infty}^{\infty} \frac{\sin^2 \frac{tx}{2}}{|t|^{1+2s}} dt \leq B^4 \int_{-\infty}^{\infty} |k(x)|^2 dx \int_{-\infty}^{\infty} \frac{\sin^2 \frac{tx}{2}}{|t|^{1+2s}} dt.$$

Changing variables in the integrals then yields

$$\int_{-\infty}^{\infty} |x|^{2s} |k_c(x)|^2 dx \int_{-\infty}^{\infty} \frac{\sin^2 \frac{y}{2}}{|y|^{1+2s}} dy \leq B^4 \int_{-\infty}^{\infty} |x|^{2s} |k(x)|^2 dx \int_{-\infty}^{\infty} \frac{\sin^2 \frac{y}{2}}{|y|^{1+2s}} dy.$$

It follows that

$$\int_{-\infty}^{\infty} |x|^{2s} |k_c(x)|^2 dx \leq B^4 \int_{-\infty}^{\infty} |x|^{2s} |k(x)|^2 dx,$$

and thus $\hat{k}_c \in H^s$. If $s = 1$, multiplying both sides of (3.2.1) by $1/t^2$ and using Fatou's lemma as $t \to 0$ yields

$$\int_{-\infty}^{\infty} |x|^2 |k_c(x)|^2 dx \leq B^4 \int_{-\infty}^{\infty} |x|^2 |k(x)|^2 dx.$$

It follows that $\hat{k}_c \in H^1$. If $s > 1$, one may apply a similar technique to derivatives of $\hat{k}_c$ and $\hat{k}$, that is to say, $\hat{k}_c'$, $\hat{k}'$, ..., $\hat{k}_c^{(n)}$ and $\hat{k}^{(n)}$, to show that $\hat{k}_c \in H^s$, where n is the integer such that $\lfloor s \rfloor \leq n \leq s$. Vice versa, $\hat{k}_c \in H^s$ also leads to $\hat{k} \in H^s$.

Now we are ready to discuss decay properties of solitary-wave solutions to the model equations under consideration.

THEOREM 3.2.1. – *Let $c > 1$ be given and suppose that the function φ satisfies the conditions* 1) $\lim_{|x| \to \infty} \varphi(x) = 0$ *and* $\varphi \in L_\infty$, *and* 2) $u(x,t) = \varphi(x - ct)$ *is a weak solution of the KdV-type equation*

$$u_t + u_x + F(u)_x - Mu_x = 0 \tag{3.2.2}$$

in the sense that

$$(\varphi, [(c-1)I + M]g') - (F(\varphi), g') = 0$$

for any $g \in C_c^\infty$. Suppose the symbol α of the dispersion operator M satisfies the smoothness condition $\hat{k} = 1/(1+\alpha) \in H^s$ for some $s > 1/2$, the operator M maps C_c^∞ into L_1 and there are constants $B > 0$ and $r > 1$ such that the function F satisfies the inequality $|F(y)| \leq B|y|^r$ at least for small values of $|y|$. Moreover, it is assumed that F' exists and is bounded on any bounded subset of $\mathbb{R}$. Then φ is a classical solution of the equation

$$\varphi + M\varphi = F(\varphi)$$

and $\hat{\varphi} \in H^s$. Furthermore, if there is a $\sigma_0 > 0$ such that k satisfies the condition

$$\int_{-\infty}^{\infty} |k(x)|^2 e^{2\sigma|x|} dx < \infty,$$

for any σ with $0 < \sigma < \sigma_0$, then $\varphi(x)e^{\sigma|x|} \in L_\infty$ for any $\sigma \in [0, \sigma_0)$.

Proof. – As discussed above, it is sufficient to consider the case $c = 2$. Choose a $\psi \in C_c^\infty$ such that $\psi(x) \geq 0$, $\operatorname{supp}\psi \subset (-1,1)$, and $\int_{-\infty}^{\infty} \psi(x)\,dx = 1$. Let $\psi_\sigma(x) = \frac{1}{\sigma}\psi(\frac{x}{\sigma})$, $\varphi_\sigma = \psi_\sigma * \varphi$ and $F(\varphi)_\sigma = F(\varphi) * \psi_\sigma$. For any $g \in C_c^\infty$, it follows from the hypotheses on φ and the fact $(\psi_\sigma \circ g)' \in C_c^\infty$ that

$$\begin{aligned}(\varphi, \psi_\sigma \circ (I+M)g') &- (F(\varphi), \psi_\sigma \circ g') \\ &= (\varphi, (I+M)(\psi_\sigma \circ g)') - (F(\varphi), (\psi_\sigma \circ g)') = 0.\end{aligned}$$

The notation $v \circ w$ is that introduced at the end of Section 2. On the other hand, it is also seen that

$$\begin{aligned}(\varphi,\, \psi_\sigma \circ (I+M)g') &- (F(\varphi),\, \psi_\sigma \circ g') \\ &= -(\varphi,\, ((I+M)\psi_\sigma') \circ g) + (F(\varphi), \psi_\sigma' \circ g) \\ &= -(((I+M)\psi_\sigma') * \varphi,\, g) + (\psi_\sigma' * F(\varphi),\, g).\end{aligned}$$

It follows that

$$(((I+M)\psi_\sigma') * \varphi,\, g) - (\psi_\sigma' * F(\varphi),\, g) = 0$$

for any $g \in C_c^\infty$, and thus

$$((I+M)\psi_\sigma') * \varphi - \psi_\sigma' * F(\varphi) = 0, \tag{3.2.3}$$

at least almost everywhere. Convolving both sides of (3.2.3) with k and using the fact that $\frac{1}{\sqrt{2\pi}}k * g = (I+M)^{-1}g$ leads to

$$\psi_\sigma' * \varphi = \frac{1}{\sqrt{2\pi}} k * \psi_\sigma' * F(\varphi). \tag{3.2.4}$$

Since $\psi_\sigma * \varphi \to 0$ and $k * \psi_\sigma * F(\varphi) \to 0$ as $|x| \to \infty$, integrating both sides of (3.2.4) from $-\infty$ to x yields

$$\psi_\sigma * \varphi(x) = \frac{1}{\sqrt{2\pi}} k * \psi_\sigma * F(\varphi)(x),$$

or, what is the same,

$$\varphi_\sigma = \frac{1}{\sqrt{2\pi}} k * F(\varphi)_\sigma.$$

Because $\lim\limits_{\sigma\to 0} \varphi_\sigma = \varphi$ and $\lim\limits_{\sigma\to 0} F(\varphi)_\sigma = F(\varphi)$, it transpires upon taking the limit as $\sigma \to 0$ of both sides of the above identity that $\varphi = \frac{1}{\sqrt{2\pi}} k * F(\varphi)$.

Then it follows from Corollary 3.1.3 that $\hat{\varphi} \in H^s$ and $F(\varphi) \in L_1 \cap L_2$ and hence it is concluded that $M\varphi = F(\varphi) - \varphi \in L_2$. Since F and φ are continuous, also as a consequence of Corollary 3.1.3, it must be that $M\varphi = F(\varphi) - \varphi$ is continuous, and thus φ is a classical solution of $\varphi + M\varphi = F(\varphi)$. The other results are a direct consequence of Corollary 3.1.4. □

The discussion of solitary-wave solutions of RLW-type equations and Schrödinger-type equations follows lines sufficiently similar to those just enunciated that we content ourselves with summary statements of the outcome.

THEOREM 3.2.2. – *Suppose that the function* φ *satisfies the conditions* 1) $\varphi \in L_\infty$ *and* $\lim_{|x|\to\infty} \varphi(x) = 0$, *and* 2) *for some constant* $c > 1$, $u(x,t) = \varphi(x - ct)$ *is a weak solution of the RLW-type equation*

$$u_t + u_x + F(u)_x + Mu_t = 0,$$

in the sense that

$$(\varphi, ((c-1)I + cM)g') - (F(\varphi), g') = 0$$

for any $g \in C_c^\infty$. *Suppose the symbol* α *of the dispersion operator* M *satisfies the smoothness condition* $\hat{k} = 1/(1+\alpha) \in H^s$ *for some* $s > 1/2$, *the operator* M *maps* C_c^∞ *into* L_1, *and there are constants* $B > 0$ *and* $r > 1$ *such that the function* F *satisfies the inequality* $|F(y)| \le B|y|^r$ *at least for small values of* $|y|$. *Moreover, it is assumed that* F' *exists and is bounded on any bounded subset of* $\mathbb{R}$. *Then* φ *is a classical solution of the equation*

$$((c-1)I + cM)\varphi = F(\varphi)$$

and $\hat{\varphi} \in H^s$. *Furthermore, if* $\hat{k}(\xi)$ *is an analytic function on the strip*

$$\{z \in \mathbb{C} : |\Im z| < \sigma_0\}$$

in the complex plane and

$$\sup_{|\eta|<\sigma} \int_{-\infty}^{\infty} |\hat{k}(\xi + i\eta)|^2 d\xi < \infty$$

for any σ *with* $0 < \sigma < \sigma_0$, *then*

$$\sup_{x\in\mathbb{R}} |\varphi(x)| e^{\sigma|x|} < \infty$$

for the same range of σ.

THEOREM 3.2.3. – *Suppose that the real-valued function* ϕ *satisfies the conditions 1)* $\phi \in L_\infty$ *and* $\lim_{|x|\to\infty} \phi(x) = 0$, *and 2) for some* $\Omega > 0$, $u(x,t) = e^{i\Omega t}\phi(x)$ *is a weak solution of the Schrödinger-type equation*

$$iu_t - Mu + F(|u|)u = 0$$

(or for some $\omega > \frac{1}{4}\theta^2$, $u(x,t) = e^{i(\theta x/2+(\omega-\frac{1}{2}\theta^2)t)}\phi(x-\theta t)$ is a weak solution of the Schrödinger-type equation) in the sense that for any $g \in C_c^\infty$,

$$(\phi, (\Omega I + M)g) - (F(|\phi|)\phi, g) = 0.$$

$$((\phi, ((\omega - \theta^2/2)I + \tilde{M})g) - (\phi, i\theta g') - (F(|\phi|)\phi, g) = 0),$$

where the symbol $\alpha \geq 0$ of the dispersion operator satisfies the condition $\hat{k} = 1/(1+\alpha) \in H^s$ (respectively, the symbol $\alpha(\xi - \theta/2)$ of the dispersion operator $\tilde{M}$ satisfies the conditions $\alpha(\xi) = \xi^2 + \beta(\xi)$ with $\beta(\xi) \geq 0$ and $\hat{k} = 1/(1+\alpha(\xi-\theta/2)+\theta\xi) \in H^s$) for some $s > 1/2$. Suppose also that the dispersion operator M (respectively, $\tilde{M}$) maps C_c^∞ to L_1, and that the function F is bounded on bounded sets and satisfies the inequality $|F(y)| \leq B|y|^r$ for all sufficiently small $y \in \mathbb{R}$ for some constants $B > 0$ and $r > 1$. Then ϕ is a classical solution of the equation

$$(\Omega I + M)\phi = F(|\phi|)\phi$$

(respectively, the equation

$$((\omega - \theta^2/2)I + \tilde{M})\phi + i\theta\phi' = F(|\phi|)\phi)$$

and $\hat{\phi} \in H^s$. Furthermore, if there is a $\sigma_0 > 0$ such that k satisfies the condition

$$\int_{-\infty}^{\infty} |k(x)|^2 e^{2\sigma|x|} dx < \infty$$

for any σ with $0 < \sigma < \sigma_0$, then $\varphi(x)e^{\sigma|x|} \in L_\infty$ for the same range of values of σ.

Remark. – When considering the analyticity of solitary-wave solutions of model equations (1.1), (1.2) and (1.3) in [14], it was assumed that the solitary-wave solution φ, its derivative φ' and $M\varphi$ were all elements of L_2. As a matter of fact, if the dispersion operator M satisfies the conditions in Theorems 3.2.1, 3.2.2 and 3.2.3 with the growth condition $\alpha(\xi) \geq A|\xi|^m$ for some constants $A > 0$ and $m \geq 1$, then it is inevitable that φ, φ' and $M\varphi$ are L_2-functions. Indeed, it was shown that φ, $M\varphi \in L_2$ in Theorem 3.2.1, and then $\varphi' \in L_2$ is simply a consequence of the growth condition $\alpha(\xi) \geq A|\xi|^m$ with $m \geq 1$. A considerable number of KdV-type equations, RLW-type equations and Schrödinger-type equations fall under the aegis of this assumption. For example, if the symbol $\alpha(\xi)$ takes the form $\alpha(\xi) = \sum_{k=1}^{N} a_k|\xi|^{r_k}$ for some constants $a_k > 0$, $r_k \geq 1$ and an integer $N > 0$, then M maps C_c^∞ to L_1 and $\hat{k} = 1/(1+\alpha) \in H^s$ for some $s > 1/2$. (Indeed, any symbol α which is locally absolutely continuous and such that, along with its derivative, is everywhere bounded by a polynomial has the property of mapping C_c^∞ to L_1. This follows since $\widehat{M\phi} = \alpha\hat{\phi}$ and the right-hand side plainly lies in H^1 if $\phi \in C_c^\infty$.) The model equations corresponding to such symbols include the Benjamin-Ono equation and the KdV equation (1.7).

It is worthwhile comparing the results of our theory for the decay of solitary waves with exact results in some cases where the solitary wave is known explicitly. We start by applying Theorems 3.1.5 and 3.2.1 to solitary-wave solutions of the the Benjamin-Ono equation

$$u_t + u_x + 2uu_x - Hu_{xx} = 0,$$

where H is the Hilbert transform defined by $H\varphi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(t)}{x-t}\,dt$. For this case, the solitary wave φ solves the integral equation $\varphi = \frac{1}{\sqrt{2\pi}}k_c * \varphi^2$ and decays at exactly the same rate as the integral kernel k_c, where $\hat{k}_c(\xi) = 1/(c-1+|\xi|)$. The reason is that k_c can be expressed as the integral

$$k_c(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{ye^{-|x|y}}{(c-1)^2+y^2}\,dy$$

and consequently

$$\lim_{|x|\to\infty} x^2 k_c(x) = \lim_{|x|\to\infty}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\eta e^{-\eta}}{(c-1)^2+\eta^2/x^2}\,d\eta = \sqrt{\frac{2}{\pi}}\frac{1}{(c-1)^2} = L.$$

It follows from Theorem 3.1.5 that

$$\lim_{|x|\to\infty} x^2\varphi^2(x) = \frac{L}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi^2(x)\,dx = \frac{1}{\pi(c-1)^2}\int_{-\infty}^{\infty}\varphi^2(x)\,dx. \tag{3.2.5}$$

Since the solitary-wave solutions of the Benjamin-Ono equation may be written in closed form as

$$\varphi(x) = \frac{2(c-1)}{1+(c-1)^2x^2},$$

simple calculations show that

$$\lim_{|x|\to\infty} x^2\varphi^2(x) = \frac{2}{c-1} \quad\text{and}\quad \int_{-\infty}^{\infty}\varphi^2(x)\,dx = 2\pi(c-1).$$

Thus in this case, (3.2.5) is directly verified. Notice that k_c can also be expressed as

$$k_c(x) = -\frac{1}{\sqrt{2\pi}}\ln[(c-1)^2x^2] + \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\ln[(c-1)^2x^2+\eta^2]e^{-\eta}d\eta,$$

showing clearly that it has a logarithmic singularity at $x = 0$. However, the solitary-wave solution φ is analytic in the strip $\{z \in C : |\Im z| < 1/(c-1)\}$ due to the algebraic decay property of k_c and the growth condition on $1/\hat{k}_c = c - 1 + |\xi|$.

Consider now the example of solitary-wave solutions of the generalized KdV equation

$$u_t + u_x + u^p u_x + u_{xxx} = 0,$$

where p is a positive number. It follows from Corollary 3.1.4 that any solitary-wave solution $\varphi(x-ct)$ with $c > 1$ decays exponentially since φ satisfies the equation $\varphi = \frac{1}{\sqrt{2\pi}}\mathcal{K}_c * \varphi^{p+1}/(p+1)$, where $\mathcal{K}_c(x) = \sqrt{\frac{\pi}{2}}\frac{e^{-\sqrt{c-1}|x|}}{\sqrt{c-1}}$ a kernel whose Fourier transform is $\hat{\mathcal{K}}_c(\xi) = 1/(c-1+\xi^2)$. Applying Theorem 3.1.6 leads to the conclusion

$$\begin{aligned}\lim_{x\to\pm\infty} e^{\sqrt{c-1}|x|}\varphi(x) &= \frac{1}{\sqrt{2\pi}(p+1)}\lim_{x\to\pm\infty} e^{\sqrt{c-1}|x|}\mathcal{K}_c(x)\int_{-\infty}^{\infty} e^{\pm\sqrt{c-1}\,t}\varphi^{p+1}(t)\,dt \\ &= \frac{1}{2\sqrt{c-1}\,(p+1)}\int_{-\infty}^{\infty} e^{\pm\sqrt{c-1}\,t}\varphi^{p+1}(t)\,dt.\end{aligned} \tag{3.2.6}$$

Since these solitary-wave solutions have the exact form

$$\varphi(x) = \sqrt[p]{\frac{(p+1)(p+2)(c-1)}{2}}\,\mathrm{sech}^{2/p}\left(\frac{\sqrt{c-1}}{2}px\right), \tag{3.2.7}$$

it is easy to see that

$$\lim_{|x|\to\infty} e^{\sqrt{c-1}|x|}\varphi(x) = \sqrt[p]{2(p+1)(p+2)(c-1)}$$

and

$$\int_{-\infty}^{\infty} e^{\sqrt{c-1}t}\varphi^{p+1}(t)\,dt = \int_{-\infty}^{\infty} e^{-\sqrt{c-1}t}\varphi^{p+1}(t)\,dt$$
$$= \frac{[2(p+1)(p+2)(c-1)]^{1+1/p}}{\sqrt{c-1}\,(p+2)}.$$

Equation (3.2.6) is again confirmed.

Attention is turned to solitary-wave solutions of the full, two-dimensional Euler equations. It was shown in the work of Friedrichs and Hyers [8] that when the Froude number $F = c/\sqrt{gh}$ is greater than but close to one, solitary-wave solutions of the Euler equations exist and are near in function space to solitary-wave solutions of the KdV equation as expressed in (1.8) or (3.2.7) with $p = 1$. As mentioned above, the KdV-solitary waves certainly decay exponentially away from their crest. Craig and Sternberg [7] have discussed the exponential decay property of the function $y(\xi,\eta)$ mentioned in the introduction corresponding to a solitary-wave solution of the Euler equations without the restriction that the Froude number lie near 1. Here, we shall apply Theorem 3.1.2 and Corollaries 3.1.3 and 3.1.4 to the solitary-wave solutions $\omega(\phi)$ of Equation (1.10) to show that $\omega(\phi)$ decays exponentially. From this it also follows that the Fourier transform $\hat{\omega}$ of ω has an analytic extension to a strip in the complex plane.

Henceforth, the following notation will be adopted from the paper of Benjamin *et al.* [5] (see (1.10)):

$$F_\gamma\omega(\phi) = \frac{\gamma\sin\omega(\phi)}{1+3\gamma\int_0^\phi \sin\omega(\tau)d\tau}.$$

THEOREM 3.2.4. – *Let $\omega(\phi)$ be a solitary-wave solution of Equation* (1.10) *which satisfies the conditions:*

$$\omega(\phi) = -\omega(-\phi) \ \textit{for any}\ \phi\in\mathbb{R} \ \textit{and}\ 0\le\omega(\phi)\le\frac{\pi}{2} \ \textit{for any}\ \phi\ge 0.$$

Then there exists a constant ν_0, such that

$$\sup_{\phi\in\mathbb{R}} |\omega(\phi)|e^{\nu|\phi|} < \infty$$

for any ν with $\nu < \nu_0$.

Proof. – In the next section, we shall prove that if ω satisfies the hypotheses of this theorem, then ω, $\sin\omega$ and $F_\gamma\omega$ all lie in $L_1 \cap L_2$. It will then follow that $\omega(\phi)$ is a continuous function on $\mathbb{R}$ with $\lim_{|\phi|\to\infty} \omega(\phi) = 0$ since the integral kernel k in (1.10) is in both L_1 and L_2, and $\lim_{|\phi|\to\infty} k(\phi) = 0$.

Noticing that as $|\phi| \to \infty$, $\frac{\gamma}{1+3\gamma\int_0^\phi \sin\omega(t)dt} \to \frac{\gamma}{1+3\gamma\int_0^\infty \sin\omega(t)dt} = \mu$, where $\mu = 1/F^2 < 1$ (see [5]), we rewrite the right-hand side of Equation (1.10) as $k * F_\gamma\omega = k * \mu\omega + k * G_\gamma\omega$, where

$$G_\gamma\omega = F_\gamma\omega - \mu\omega = \frac{\gamma(\sin\omega - \omega)}{1 + 3\gamma\int_0^\phi \sin\omega(t)dt} + \frac{3\gamma^2\,\omega\int_{|\phi|}^\infty \sin\omega(t)dt}{(1+3\gamma\int_0^\phi \sin\omega(t)dt)(1+3\gamma\int_0^\infty \sin\omega(t)dt)}. \tag{3.2.8}$$

Then taking the Fourier transform of both sides of Equation (1.10) yields $\hat{\omega} = \mu\hat{k}\hat{\omega} + \hat{k}\widehat{G_\gamma\omega}$, or, solving for $\hat{\omega}$,

$$\hat{\omega} = \frac{\hat{k}}{1-\mu\hat{k}}\widehat{G_\gamma\omega} = \frac{\sinh\xi}{\xi\cosh\xi - \mu\sinh\xi}\widehat{G_\gamma\omega}(\xi).$$

Since $\mu < 1$, the meromorphic function $\frac{\sinh\xi}{\xi\cosh\xi - \mu\sinh\xi}$ has countably many poles located at points $\xi = i\eta$ where the real numbers $\eta \neq 0$ satisfy the relation $\eta = \mu\tan\eta$. An application of the residue theorem shows that its inverse Fourier transform can be expressed as

$$h(\phi) = \sqrt{2\pi}\sum_{n=1}^\infty \frac{\tan\eta_n}{\eta_n\tan\eta_n + \mu - 1}e^{-\eta_n|\phi|},$$

for any $|\phi| > 0$, where $\{\eta_n\}_{n=1}^\infty$ comprise the poles of $\hat{h}$ on the positive imaginary axis, numbered so that $\eta_n < \eta_{n+1}$ for $n = 1, 2, 3, \cdots$. Therefore, ω satisfies the following, equivalent integral equation:

$$\omega(\phi) = \frac{1}{\sqrt{2\pi}} h * G_\gamma\omega(\phi). \tag{3.2.9}$$

Since

$$|G_\gamma\omega| \leq \gamma|\sin\omega - \omega| + 3\gamma^2|\omega|\int_{|\phi|}^\infty \sin\omega(t)\,dt,$$

$$|\sin\omega - \omega| \leq \omega^2 \quad\text{and}\quad \lim_{|\phi|\to\infty}\int_{|\phi|}^\infty \sin\omega(t)\,dt = 0,$$

with a slight modification in the proof of Theorem 3.1.2 and Corollary 3.1.4, one may deduce that

$$\omega(\phi)e^{\nu|\phi|} \in L_\infty \cap C(\mathbb{R}),$$

for any ν with $0 < \nu \leq \eta_1$, where $\eta_1 > 0$ is the solution closest to the origin of the equation $\eta = \mu \tan \eta$. It also follows from Theorem 3.1.6 that

$$\lim_{\phi \to \pm\infty} e^{\eta_1 |\phi|} \omega(\phi) = \frac{\tan \eta_1}{\eta_1 \tan \eta_1 + \mu - 1} \int_{-\infty}^{\infty} e^{\pm \eta_1 t} G_\gamma \omega(t)\, dt,$$

which implies that $\nu_0 \geq \eta_1$. □

One consequence of Theorem 3.2.4 is the analyticity of the Fourier transforms $\hat{\omega}$, $\widehat{\sin \omega}$, $\widehat{F_\gamma \omega}$ and $\widehat{G_\gamma \omega}$ stated in the following corollary.

COROLLARY 3.2.5. – *Under the conditions of Theorem* 3.2.4, *the Fourier transforms* $\hat{\omega}$, $\widehat{\sin \omega}$, $\widehat{F_\gamma \omega}$ *and* $\widehat{G_\gamma \omega}$ *of the functions* ω, $\sin \omega$, $F_\gamma \omega$ *and* $G_\gamma \omega$, *respectively, have analytic extensions to the strip* $\{\xi + i\eta, |\eta| < \nu_0\}$.

Proof. – Because $\omega(\phi) e^{\nu|\phi|} \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$, $\omega(\phi) e^{\nu|\phi|} \in L_2(\mathbb{R})$ for any $0 < \nu < \nu_0$. Moreover, $|\sin \omega| \leq |\omega|$, $|F_\gamma \omega| \leq |\gamma \omega|$ and $|G_\gamma \omega| \leq |F_\gamma \omega| + \mu|\omega|$. The conclusion follows from the Paley-Wiener Theorem [19]. □

Remark. – One may also use Craig and Sternberg's method to show this decay property of solitary-wave solutions to the full Euler equations. The results are here obtained as an easy corollary to our general theory about decay of solutions of convolution equations of the form $f = k * G(f)$.

The decay property of solitary-wave solutions to the Euler equations demonstrates the important role played by the nonlinearity, and in particular by the inequality

$$|G(u)| \leq M|u|^r \tag{3.2.10}$$

for some $r > 1$, which is satisfied by the convolution equations under consideration. Notice that a solitary-wave solution ω of the Euler equations satisfies both Equations (1.10) and (3.2.8). It follows from Craig and Sternberg's result [7] that ω decays exactly at the same order $e^{-\eta_1|\phi|}$ as the kernel h in (3.2.8), with $0 < \eta_1 < \pi/2$, but apparently more slowly than the kernel k in (1.10), whose decay has the asymptotic form $e^{-\pi|\phi|/2}$. In Equation (1.10), the nonlinear term $F_\gamma \omega$ satisfies the inequality $|F_\gamma \omega| \leq \gamma|\omega|$, while the nonlinear term $G_\gamma \omega$ in Equation (3.2.8) possesses the property $|G_\gamma \omega| \leq M|\omega|^2$ for some constant $M > 0$ for sufficiently small values of ω. Thus the super-linear condition (3.2.10) is needed, together with the decay condition imposed on the kernel k, in order that solutions of $f = k * G(f)$ evanesce at infinity at least as rapidly as the kernel. Without Inequality (3.2.10), the results obtained in this section may not be valid. An additional matter worth mention is that larger values of $r > 1$ in (3.2.10) do not necessarily imply a higher order of decay in the face of a fixed integral kernel k. Solitary-wave solutions of the generalized KdV equations provide a clear example. They all decay at the same order regardless of the value of $p > 0$ appearing in the equation.

4. Analyticity of solitary waves

This section is devoted to extending the result on analyticity of solitary-wave solutions developed in [14] so that the outcome will not only apply to more general model equations, but also to Equation (1.10) (repeated here for convenience)

$$\omega(\phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(\phi - t) \frac{\gamma \sin \omega(t)}{1 + 3\gamma \int_0^t \sin \omega(\tau) d\tau} dt \tag{4.0.1}$$

for solitary-wave solutions of the full, two dimensional Euler equations. For the Euler equations, once the free-surface variable is known to be given by a function which is the restriction of an analytic function in a strip, it is natural to ask about the analyticity properties of the velocity potential ϕ and the stream function ψ. These will be seen to have an analytic extension to an open set in 2-dimensional complex space $\mathbb{C}^2$. As a simple consequence, it will be ascertained that all the streamlines are the restrictions of analytic functions.

4.1. Regularity of solutions to nonlinear convolution equations

In [14], solitary-wave solutions of model equations of the form depicted in (1.1), (1.2) and (1.3) were shown to be the restriction to the real axis of functions analytic in a strip in the complex plane. This was accomplished under the assumption of a homogeneous nonlinearity $f(u)_x$ for (1.1) or (1.2) and $f(|u|)u$ for (1.3), where $f(z) = z^{p+1}$, say, with p a positive integer. The general aim here is to extend the range of this result to include a much broader class of analytic nonlinearities. This will be accomplished by reconsidering the associated nonlinear convolution equations $f = k * G(f)$.

The following three technical lemmas prepare the way for the further study of these convolution equations.

LEMMA 4.1.1. – *Let f and g be infinitely differentiable functions defined on some open interval $I \in \mathbb{R}$. For $x \in I$, denote by y the value $g(x)$. For any integer $n \geq 2$, we have*

$$\begin{aligned} \frac{d^n f(g(x))}{dx^n} &= y^{(n)} f'(y) \\ &\quad + \sum_{s=2}^{n} \frac{f^{(s)}(y)}{s!} \sum\nolimits^{(s,n)} \binom{n}{j_1, \cdots, j_{s-2}, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots y^{(j_{s-1})}, \end{aligned}$$

where

$$y^{(k)} = \frac{d^k g(x)}{dx^k},$$

$$\sum\nolimits^{(s,n)} = \sum_{j_1=s-1}^{n-1} \sum_{j_2=s-2}^{j_1-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}-1}$$

and

$$\binom{n}{j_1, \cdots, j_{s-2}, j_{s-1}} = \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}}.$$

The proof of Lemma 4.1.1 is tedious, but straightforward. A sketch is provided in the Appendix. The notations $\sum^{(s,n)}$ and $\begin{pmatrix} & n & \\ j_1, \cdots, j_{s-2}, j_{s-1} \end{pmatrix}$ defined in Lemma 4.1.1 will be used throughout this section.

LEMMA 4.1.2. – *For any integer $n \geq 2$ and any integer s with $2 \leq s \leq n$, we have that*

$$\sum\nolimits^{(s,n)} \binom{n}{j_1, \ldots, j_{s-1}} (n-j_1)^{n-j_1-1} (j_1 - j_2)^{j_1-j_2-1} \ldots$$
$$\cdots (j_{s-2} - j_{s-1})^{j_{s-2}-j_{s-1}-1} {j_{s-1}}^{j_{s-1}-1} = \frac{s n! n^{n-s-1}}{(n-s)!}. \tag{4.1.1}$$

Proof. – First remark that the identity (see [20, p. 23])

$$\sum_{j=0}^{m} \binom{m}{j} (x+j)^{j-1} (y+m-j)^{m-j-1} = \left(\frac{1}{x} + \frac{1}{y}\right)(x+y+m)^{m-1}$$

of Abel implies that

$$\sum_{j=0}^{m-1} \binom{m}{j} (x+j)^{j-1} (m-j)^{m-j-1} = \frac{(x+m)^{m-1}}{x} + (m-1)(x+m)^{m-2} \tag{4.1.2}$$

for any $m \geq 1$ and any $x \neq 0$.

Consider the left-hand side of (4.1.1) for $s = 2$ and an integer $n \geq 2$. Let $r = j - 1$ and then apply (4.1.2) to obtain:

$$\begin{aligned} &\sum_{j=1}^{n-1} \binom{n}{j} (n-j)^{n-j-1} j^{j-1} \\ &\quad = n \sum_{r=0}^{n-2} \binom{n-1}{r} (n-1-r)^{n-2-r} (1+r)^{r-1} \\ &\quad = n\big[n^{n-2} + (n-2)n^{n-3}\big] = \frac{2n! n^{n-3}}{(n-2)!}. \end{aligned}$$

An induction on s is used to complete the proof. Suppose that there exists an integer s with $s \geq 2$, such that the equality

$$\sum_{j_1=s-1}^{l-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{l}{j_1, \ldots, j_{s-1}} (l-j_1)^{l-j_1-1} (j_1 - j_2)^{j_1-j_2-1} \ldots$$
$$\cdots (j_{s-2} - j_{s-1})^{j_{s-2}-j_{s-1}-1} (j_{s-1})^{j_{s-1}-1} = \frac{s\, l!\, l^{l-s-1}}{(l-s)!}$$

holds for any integer l satisfying $s \leq l$. Let $r = s + 1$ and apply the induction hypothesis to the left-hand side of (4.1.1) for an integer n with $n \geq s + 1$ to adduce

$$\begin{aligned}(4.1.3)\quad &\sum_{j_1=s}^{n-1} \binom{n}{j_1} (n - j_1)^{n-j_1-1} \sum_{j_2=s-1}^{j_1-1} \cdots \sum_{j_s=1}^{j_{s-1}-1} \binom{j_1}{j_2, \ldots, j_s} (j_1 - j_2)^{j_1-j_2-1} \cdots \\ &\cdots (j_{s-1} - j_s)^{j_{s-1}-j_s-1} j_s^{j_s-1} \\ &= \sum_{j_1=s}^{n-1} \binom{n}{j_1} (n - j_1)^{n-j_1-1} \frac{s\, j_1!\, j_1^{j_1-s-1}}{(j_1 - s)!}.\end{aligned}$$

Letting $j = j_1 - s$ and applying (4.1.2) to (4.1.3) yields

$$\begin{aligned}&\frac{sn!}{(n-s)!} \sum_{j=0}^{n-s-1} \binom{n-s}{j} (s+j)^{j-1} (n-s-j)^{n-s-j-1} \\ &= \frac{sn!}{(n-s)!} \left[\frac{n^{n-s-1}}{s} + (n-s-1) n^{n-s-2} \right] = \frac{(s+1) n! n^{n-s-2}}{(n-s-1)!}.\end{aligned}$$

It is thereby proved that identity (4.1.1) holds for all integers n and s with $2 \leq s \leq n$. □

LEMMA 4.1.3. – *The identity*

$$\begin{aligned}(4.1.4)\quad &\sum_{0}^{n-2} \binom{n}{j} (x+j)^{j-1} (n-j-1)^{n-j-2} \\ &= \frac{(x+n-1)^{n-2}}{x} + (x+n-1)^{n-1} - (x+n)^{n-1} + \\ &+ n(n-2)(x+n-1)^{n-3} + (n-1)(x+n-1)^{n-2}\end{aligned}$$

holds for any integer $n \geq 2$ *and any number* $x > 0$.

Proof. – It follows from another version of Abel's identities [20], namely

$$\begin{aligned}\sum_{j=0}^{n} \binom{n}{j} (x+j)^{j-1} (y+n-j)^{n-j-2} &= \left(\frac{1}{x} + \frac{1}{1+y} \right) (x+y+n)^{n-2} \\ &+ \frac{(x+y+n)^{n-1}}{y^2} - \frac{(n-1)(x+y+n)^{n-2}}{y(1+y)},\end{aligned}$$

that

$$\begin{aligned}\sum_{j=0}^{n-2} \binom{n}{j} (x+j)^{j-1} (y+n-j)^{n-j-2} &= \left(\frac{1}{x} + \frac{1}{1+y} \right) (x+y+n)^{n-2} \\ &+ \frac{(x+y+n)^{n-1}}{y^2} - \frac{(n-1)(x+y+n)^{n-2}}{y(1+y)} - \frac{(x+n)^{n-1}}{y^2} - \frac{n(x+n-1)^{n-2}}{1+y}.\end{aligned}$$

where y is any real number with $-1 < y < 0$. Taking the limit of both sides of the above identity as $y \to -1$ leads to (4.1.4). □

The way is cleared for a discussion of regularity of solutions to nonlinear convolution equations, which is the subject of the next theorem. To establish the result in view, the radius of convergence r of the Taylor series expansion of the solution f to the given convolution equation is estimated at every point $x \in \mathbb{R}$. It will be demonstrated that there is a constant $\sigma_0 > 0$, independent of x, such that $r \geq \sigma_0$, which implies f to have an analytic extension to the strip $\{z \in \mathbb{C} : |\Im z| < \sigma_0\}$.

THEOREM 4.1.4. – *Suppose that f is a solution of the convolution equation $f = k * G(f)$ such that $f \in L_2 \cap L_\infty$ and $\lim_{|x|\to\infty} f(x) = 0$. If the Fourier transform $\hat{k}$ of the integral kernel k satisfies the decay condition $|\hat{k}(\xi)| \leq A_1/(1 + A_2|\xi|^m)$ for some constants A_1, $A_2 > 0$ and $m \geq 1$, and $G(z)$ is an entire function defined on the complex plane $\mathbb{C}$ with $G(0) = 0$, then there exists a constant $\sigma_0 > 0$ such that f has an analytic extension to the strip $\{z \in \mathbb{C} : |\Im z| < \sigma_0\}$.*

Proof. – Since $G(z)$ is analytic at the origin with $G(0) = 0$, there is a constant $M > 0$ such that $|G(z)| \leq M|z|$ for all z with $|z|$ sufficiently small. This fact together with the hypothesis $f \in L_2 \cap L_\infty$ implies that $G(f) \in L_2 \cap L_\infty$.

Next, it is shown that f, $G(f) \in H^\infty$. It follows from the Plancherel theorem that for any $g \in C_c^\infty(\mathbb{R})$,

$$(f, g') = (\hat{f}, \widehat{g'}) = (\widehat{k * G(f)}, \widehat{g'}) = \int_{-\infty}^{\infty} \hat{k}(\xi)\widehat{G(f)}(\xi)i\xi\overline{\hat{g}(\xi)}d\xi. \tag{4.1.5}$$

Because of the decay condition satisfied by $\hat{k}$, there is a constant A_3 such that $\left|\xi\hat{k}(\xi)\right| \leq A_3$, and thus

$$|(f, g')| \leq A_3 \int_{-\infty}^{\infty} \left|\widehat{G(f)}(\xi)\hat{g}(\xi)\right| d\xi \leq A_3 \left\|\widehat{G(f)}\right\|_2 \|\hat{g}\|_2 = A_3\|G(f)\|_2\|g\|_2.$$

This implies that f' exists in the sense of distribution, and that $f' \in L_2(\mathbb{R})$, or what is the same $f \in H^1(\mathbb{R})$, with $\|f'\|_2 \leq A_3\|G(f)\|_2$.

Because $f' \in L_2$ and $\frac{dG(f(x))}{dx} = G'(f(x))f'(x)$, we have $(G(f))' \in L_2$ also. Thus (4.1.5) and the Parseval theorem imply that

$$f' = k * (G(f))'.$$

It follows that f, $G(f) \in H^1$ and f', $(G(f))' \in L_\infty$.

To prove that f, $G(f) \in H^m$, one may argue inductively. Suppose that there is an integer $m \geq 1$ such that f, $G(f) \in H^m$, and

$$f^{(j)} = k * (G(f))^{(j)}, \tag{4.1.6}$$

for all integers j with $0 \leq j \leq m$. Then for any $g \in C_c^\infty$ and for $j = m$, (4.1.6) and the Plancherel theorem lead to the relation

$$\begin{aligned}(f^{(m)}, g') &= (k * (G(f))^{(m)}, g') \\ &= ((k * (G(f))^{(m)})^{\widehat{}}, \widehat{g'}) = \int_{-\infty}^{\infty} \hat{k}(\xi)(\widehat{G(f)})^{(m)}(\xi)i\xi\overline{\hat{g}(\xi)}d\xi.\end{aligned}$$

Use the inequality $|\xi\hat{k}(\xi)| \leq A_3$ again to obtain the norm estimate:

$$\begin{aligned}\left|\left(f^{(m)},\, g'\right)\right| &\leq A_3 \int_{-\infty}^{\infty} |\hat{g}(\xi)| \left|(\widehat{G(f)})^{(m)}(\xi)\right| d\xi \\ &\leq A_3 \|\hat{g}\|_2 \left\|(\widehat{G(f)})^{(m)}\right\|_2 = A_3 \|g\|_2 \left\|(G(f))^{(m)}\right\|_2.\end{aligned}$$

This means that $f^{(m+1)}$ exists in the sense of distribution and $f^{(m+1)} \in L_2(\mathbb{R})$ with $\|f^{(m+1)}\|_2 \leq A_3 \|(G(f))^{(m)}\|_2$.

Now compute $(G(f))^{(n)}$ by applying Lemma 4.1.1 thusly:

$$\begin{aligned}\text{(4.1.7)} \quad \frac{d^n G(f(x))}{dx^n} &= f^{(n)} G'(f) \\ &+ \sum_{s=2}^{n} \frac{G^{(s)}(f)}{s!} \sum\nolimits^{(s,n)} \binom{n}{j_1, \cdots, j_{s-2}, j_{s-1}} f^{(n-j_1)} f^{(j_1-j_2)} \cdots f^{(j_{s-1})}\end{aligned}$$

for any integer $n \geq 3$. Formula (4.1.7) applied when $n = m+1$ shows that $(G(f))^{(m+1)}$ may be expressed in terms of derivatives of G and f with orders not greater than $m+1$. Hence the induction hypotheses, the boundedness of $f^{(j)}$ which follows from $|f^{(j)}| \leq \|k\|_2 \|(G(f))^{(j)}\|_2$ for any integer j with $0 \leq j \leq m$ and the fact $f^{(m+1)} \in L_2$ lead to the conclusion $(G(f))^{(m+1)} \in L_2$. By induction, we adduce that f and $G(f) \in H^\infty$ and that the following two relations hold for any integer $m \geq 1$,

$$\text{(4.1.8)} \qquad f^{(m)} = k * (G(f))^{(m)} \qquad \text{and} \qquad \|f^{(m)}\|_2 \leq A_3 \|(G(f))^{(m-1)}\|_2.$$

Next we estimate the L_2-norm of $(G(f))^{(n)}$ for $n = 1, 2, \cdots$. Since f is a continuous and bounded function defined on $\mathbb{R}$, its range $R(f)$ is a bounded subset of $\mathbb{R}$ and hence of $\mathbb{C}$. Let γ be a closed Jordan curve whose interior contains $R(f)$ for which $d = \operatorname{dist}(\gamma, R(f)) = \inf_{\substack{x\in\mathbb{R}\\ \xi_1\in\gamma}} |\xi_1 - f(x)| > \|k\|_2$. Let $M_1 = \max_{\xi\in\gamma}\{|G(\xi)|\}$. The Cauchy formula for the nth derivative, applied to the entire function G at a point $f(x)$ implies

$$G^{(n)}(f(x)) = \frac{n!}{2\pi i} \int_\gamma \frac{G(\xi)}{(\xi - f(x))^{n+1}} d\xi$$

and this leads to the estimate

$$\text{(4.1.9)} \qquad \frac{|G^{(n)}(f(x))|}{n!} \leq \frac{M_1 |\gamma|}{2\pi d^{n+1}} = \frac{M_2}{d^{n+1}},$$

valid for any $x \in \mathbb{R}$, where $|\gamma|$ represents the length of γ.

We aim now to derive L_2-bounds to supplement the L_∞-bounds in (4.1.9). To this end, define two constants:

$$a_1 = \max_{n=1,2}\{\|(G(f))^{(n)}\|_2^{\frac{1}{n}}\}$$

and

$$a = \max\left\{a_1, A_3 M_2\left[\frac{1}{3d^3} + \frac{(2 + \|G(f)\|_2)\|k\|_2}{d(d - \|k\|_2)^2}\right]\right\}.$$

It is obvious that

$$\|(G(f))^{(n)}\|_2 \le a^n n^{n-1} \tag{4.1.10}$$

for $n = 1, 2$, with this definition of a. We claim (4.1.10) is valid for all positive integers. To use induction, suppose $n \ge 2$ to be such that (4.1.10) holds for any integer m with $1 \le m \le n$. Estimating $\|(G(f))^{(n+1)}\|_2$ by applying (4.1.7), (4.1.8), (4.1.9), the induction hypothesis and Lemma 4.1.2 results in the inequality

$$\begin{aligned}
\left\|(G(f))^{(n+1)}\right\|_2 &\le \frac{A_3 M_2}{d^2}\left\|(G(f))^{(n)}\right\|_2 \\
&\quad + \sum_{s=2}^{n+1} \frac{M_2}{d^{s+1}} \sum\nolimits^{(s,n+1)} \binom{n+1}{j_1, \cdots, j_{s-1}} \|f^{(n+1-j_1)}\|_2 \|f^{(j_1-j_2)} \cdots f^{(j_{s-1})}\|_\infty \\
&\le \frac{A_3 M_2}{d^2}\left\|(G(f))^{(n)}\right\|_2 + \sum_{s=2}^{n+1} \frac{A_3 M_2 \|k\|_2^{s-1}}{d^{s+1}} \sum\nolimits^{(s,n+1)} \binom{n+1}{j_1, \cdots, j_{s-2}, j_{s-1}} \\
&\quad \cdot \left\|(G(f))^{(n-j_1)}\right\|_2 \left\|(G(f))^{(j_1-j_2)}\right\|_2 \cdots \left\|(G(f))^{(j_{s-1})}\right\|_2 \\
&\le \frac{A_3 M_2}{d^2} a^n n^{n-1} + \sum_{s=2}^{n+1} \frac{A_3 M_2 \|k\|_2^{s-1}}{d^{s+1}} \sum_{j_1=s-1}^{n} \binom{n+1}{j_1} \left\|(G(f))^{(n-j_1)}\right\|_2 \\
&\quad \cdot \sum\nolimits^{(s-1,j_1)} \binom{j_1}{j_2, \cdots, j_{s-1}} a^{j_1-j_2} (j_1 - j_2)^{j_1-j_2-1} \cdots \\
&\quad \cdots a^{j_{s-2}-j_{s-1}} (j_{s-2} - j_{s-1})^{j_{s-2}-j_{s-1}-1} a^{j_{s-1}} (j_{s-1})^{j_{s-1}-1} \\
&= \frac{A_3 M_2}{d^2} a^n n^{n-1} + \sum_{s=2}^{n+1} \frac{A_3 M_2 \|k\|_2^{s-1}}{d^{s+1}} \sum_{j_1=s-1}^{n} \binom{n+1}{j_1} \\
&\quad \cdot \frac{a^{j_1} (s-1) j_1! j_1^{j_1-s}}{(j_1 - s + 1)!} \left\|(G(f))^{(n-j_1)}\right\|_2 \\
&\le \frac{A_3 M_2}{d^2} a^n n^{n-1} + \sum_{s=2}^{n+1} \frac{A_3 M_2 \|k\|_2^{s-1} (s-1)}{d^{s+1}} \left(\frac{a^n (n+1)! n^{n-s}}{(n - s + 1)!} \|G(f)\|_2 \right. \\
&\quad \left. + \sum_{j_1=s-1}^{n-1} \binom{n+1}{j_1} \frac{a^{j_1} j_1! j_1^{j_1-s}}{(j_1 - s + 1)!} a^{n-j_1} (n - j_1)^{n-j_1-1} \right).
\end{aligned} \tag{4.1.11}$$

Let $j_1 - s + 1 = j$ and use Lemma 4.1.3 to deduce

$$\begin{aligned}
&\sum_{j_1=s-1}^{n-1} \binom{n+1}{j_1} \frac{j_1! j_1^{j_1-s}}{(j_1 - s + 1)!} (n - j_1)^{n-j_1-1} \\
&= \frac{(n+1)!}{(n - s + 2)!} \sum_{j=0}^{n-s} \binom{n - s + 2}{j} (j + s - 1)^{j-1} (n - s + 2 - j - 1)^{n-s-j}
\end{aligned} \tag{4.1.12}$$

$$= \frac{(n+1)!}{(n-s+2)!}\left[\frac{n^{n-s}}{s-1} + n^{n-s+1} - (n+1)^{n-s+1} \right.$$
$$\left. + (n-s+2)(n-s)n^{n-s-1} + (n-s+1)n^{n-s}\right].$$

It follows from (4.1.11) and (4.1.12) and the definition of a that

$$\begin{aligned}
&\|(G(f))^{(n+1)}\|_2 \\
&\leq \frac{A_3 M_2 a^n}{d^2}\left[n^{n-1} + (2+\|G(f)\|_2)(n+1)!\sum_{s=2}^{n+1}\frac{(s-1)\|k\|_2^{s-1}n^{n-s}}{d^{s-1}(n-s+1)!}\right] \\
&\leq \frac{A_3 M_2 a^n}{d^2}\left[n^{n-1} + (2+\|G(f)\|_2)(n+1)n^{n-1}\sum_{s=2}^{n+1}\frac{(s-1)\|k\|_2^{s-1}}{d^{s-1}}\right] \\
&\leq (n+1)^n A_3 M_2 a^n\left[\frac{1}{3d^2} + (2+\|G(f)\|_2)\frac{\|k\|_2}{d(d-\|k\|_2)^2}\right] \leq a^{n+1}(n+1)^n.
\end{aligned}$$

Thus Inequality (4.1.10) holds for any integer $n \geq 1$.

For any $x \in \mathbb{R}$ and any integer $n \geq 2$, (4.1.8) yields

$$\begin{aligned}
|f^{(n)}(x)|^2 &= 2\left|\int_{-\infty}^{x} f^{(n)}(t)f^{(n+1)}(t)\,dt\right| \\
&\leq 2\|f^{(n)}\|_2\|f^{(n+1)}\|_2 \leq 2A_3^2\left\|(G(f))^{(n-1)}\right\|_2\left\|(G(f))^{(n)}\right\|_2 \\
&\leq 2A_3^2 a^{2n-1}(n-1)^{n-2}n^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\left|(G(f(x)))^{(n)}\right|^2 &= 2\left|\int_{-\infty}^{x}(G(f(t)))^{(n)}(G(f(t)))^{(n+1)}\,dt\right| \\
&\leq 2\left\|(G(f))^{(n)}\right\|_2\left\|(G(f))^{(n+1)}\right\|_2 \leq 2a^{2n+1}n^{n-1}(n+1)^n.
\end{aligned}$$

In consequence, it is seen that

$$\sqrt[n]{\frac{|f^{(n)}(x)|}{n!}} \leq \sqrt[n]{\frac{\sqrt{2}A_3 a^{n-1/2}n^{n-1}}{n!}} \underset{n\to\infty}{\longrightarrow} ae$$

and

$$\sqrt[n]{\frac{|(G(f(x)))^{(n)}|}{n!}} \leq \sqrt[n]{\frac{\sqrt{2}a^{n+1/2}(n+1)^n}{n!}} \underset{n\to\infty}{\longrightarrow} ae.$$

It follows from the above two inequalities that f and $G(f)$ have Taylor series expansions about any point $x \in \mathbb{R}$ with radius of convergence $R \geq \frac{1}{ae}$, a quantity which is independent of the point x. Therefore, there exists a constant $\sigma_0 \geq 1/ae$ for which f and $G(f)$ have analytic extensions to the strip

$$S_{\sigma_0} = \{z \in \mathbb{C} : \ |\Im z| < \sigma_0\}. \qquad \square$$

One interesting consequence of Theorem 4.1.4 is that the analytic extensions of f and $G(f)$ are also L_2-functions in S_{σ_0}. This leads in turn to the conclusion stated in the next corollary, that the Fourier transforms of f and $G(f)$ decay exponentially at infinity.

COROLLARY 4.1.5. – *The Fourier transforms $\hat{f}$ and $\widehat{G(f)}$ of f and $G(f)$ satisfy the inequalities*

$$\int_{-\infty}^{\infty} \left|\hat{f}(t)\right|^2 e^{2\mu|t|}dt < \infty,$$
$$\int_{-\infty}^{\infty} \left|\widehat{G(f)}(t)\right|^2 e^{2\mu|t|}dt < \infty,$$

respectively, for any μ with $0 < \mu < \frac{1}{ae}$.

Proof. – Let $z = x + iy$ be any point satisfying $|y| \leq \mu$ with $0 < \mu < 1/ae$. Then the Taylor expansion of f at $z_0 = x$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(z - x)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(iy)^n$$

and thus

$$\left(\int_{-\infty}^{\infty} |f(x+iy)|^2 dx\right)^{\frac{1}{2}} \leq \sum_{n=0}^{\infty} \frac{|y|^n}{n!}\left(\int_{-\infty}^{\infty} \left|f^{(n)}(x)\right|^2 dx\right)^{\frac{1}{2}}$$
$$= \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_2}{n!}|y|^n \leq \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_2}{n!}\mu^n < \infty.$$

The above inequality and the Paley-Wiener theory [19, Theorem IV] imply that

$$\int_{-\infty}^{\infty} \left|\hat{f}(t)\right|^2 e^{2\mu|t|}dt < \infty.$$

is valid for any μ with $0 < \mu < \frac{1}{ae}$.

The other inequality may be verified similarly. $\square$

Remark. – In adducing analyticity of f in Theorem 4.1.4, we have actually used only two properties of G: (i) $G(0) = 0$, and (ii) G is an analytic function on an open set U containing the range $R(f)$ of the solution f for which G is continuous up to the boundary ∂U of U and

$$\operatorname{dist}(\partial U, R(f)) = \inf_{\substack{x \in \mathbb{R} \\ z \in \partial U}} |z - f(x)| > \|k\|_2.$$

This observation allows one to extend Theorem 4.1.4 in the following way.

COROLLARY 4.1.6. – *Suppose that f is a solution of the convolution equation $f = k * G(f)$ such that $f \in L_2 \cap L_\infty$ and $\lim_{|x|\to\infty} f(x) = 0$. If the Fourier transform $\hat{k}$ of the integral kernel k satisfies the decay condition $|\hat{k}(\xi)| \le A_1/(1 + A_2|\xi|^m)$ for some constants A_1, $A_2 > 0$ and $m \ge 1$, and G is an infinitely differentiable function whose domain contains the range $R(f)$ of f, having all of its derivatives bounded on $R(f)$ and satisfying the condition $G(0) = 0$, then f, $G(f) \in H^\infty$. In addition, if G is an analytic function on an open set U containing $R(f)$, G is continuous up to the boundary ∂U of U and*

$$\operatorname{dist}(\partial U, R(f)) = \inf_{\substack{x\in\mathbf{R}\\ z\in\partial U}} |z - f(x)| > \|k\|_2,$$

then there exists a constant $\sigma_0 > 0$ such that f and $G(f)$ both have analytic extensions to the strip $\{z \in \mathbb{C} : |\Im z| < \sigma_0\}$.

It is worth summarizing the overall view of solutions of the nonlinear convolution equation $f = k * G(f)$ gleaned from the preceding development.

THEOREM 4.1.7. – *Suppose that f is a solution of the convolution equation*

$$f(x) = (k * G(f))(x)$$

such that $f \in L_\infty$ and $\lim_{|x|\to\infty} f(x) = 0$. Suppose also the measurable function G satisfies the condition $|G(u)| \le M|u|^r$ for some constants $M > 0$ and $r > 1$ and all sufficiently small values of $|u|$ and the integral kernel k satisfies the condition $\hat{k} \in H^s$ for some $s > 1/2$. Then f is a bounded and continuous function with $(1 + |x|)^s f(x) \in L_2 \cap L_\infty$.

Furthermore, under the condition that the Fourier transform $\hat{k}$ of k is an analytic function on the strip $\{z \in \mathbb{C} : |\Im z| < \sigma_0\}$ satisfying

$$\sup_{|\eta|<\sigma} \int_{-\infty}^{\infty} |\hat{k}(\xi + i\eta)|^2 d\xi < \infty$$

for any σ with $0 < \sigma < \sigma_0$, then $e^{\sigma|x|} f(x) \in L_\infty$ for all such values of σ.

In addition, if $G = G(z)$ is an analytic function, satisfying the property (ii) *in the last Remark, and $|\hat{k}(\xi)| \le A_1/(1 + A_2|\xi|^m)$, where A_1, $A_2 > 0$ and $m \ge 1$ are constants, then f and $G(f)$ have analytic extensions F and $G(F)$ defined on a horizontal strip $\{z \in \mathbb{C} : |\Im z| < \nu_0\}$ for some constant $\nu_0 > 0$ with F and $G(F)$ satisfying the inequalities*

$$\sup_{|y|<\nu} \int_{-\infty}^{\infty} |F(x + iy)|^2 \, dx < \infty,$$

$$\sup_{|y|<\nu} \int_{-\infty}^{\infty} |G(F(x + iy))|^2 \, dx < \infty,$$

for any ν with $0 < \nu < \nu_0$, respectively.

As pointed out in Theorems 3.2.1, 3.2.2 and 3.2.3, solitary-wave solutions of the evolution equations under consideration can be expressed as solutions of convolution equations of the form $\varphi = k * G(\varphi)$. Therefore, the conclusions of Theorem 4.1.7 apply to these solutions when k and G satisfy the hypotheses of the theorem.

In the next subsection, it will be demonstrated that solitary-wave solutions of the Euler equations fall into this category as well.

It was proved by Amick and Toland [3] that solitary-wave solutions of the full Euler equations are real analytic functions, but the issue of how far solutions could be extended into the complex plane was not addressed. To cast light on this question, one might adopt the method Lewy used in his work [13], which is connected to the problem of local extension of a harmonic function satisfying certain boundary conditions. However, it will be seen in the next subsection that the technique just developed can also be used to tackle the issue of analyticity of these solitary-wave solutions.

4.2. Analyticity of solutions $\omega(\phi)$ to Equation (4.0.1)

In the work of Benjamin *et al.* [5], it was shown that Equation (4.0.1) has solitary-wave solutions $\omega(\phi)$ which are odd functions on $\mathbb{R}$ and non-negative for $\phi \geq 0$. Moreover, because $|\omega(\phi)|$ is bounded by the L_2-function $|k * (1/\phi)|$, $\omega(\phi) < \pi/3$ for any $\phi \geq 0$ and $\lim_{|\phi| \to \infty} \omega(\phi) = 0$. Here, we intend to show in Theorem 4.2.2 that a solution ω of Equation (4.0.1) is necessarily an element of $L_1 \cap L_2$, as long as ω is an odd function and $0 \leq \omega \leq \pi/2$ on the interval $(0, \infty)$. We will then be able to conclude that ω satisfies the conditions imposed on the solutions in Theorem 4.1.4.

We begin by exposing a few basic properties of the kernel k.

LEMMA 4.2.1. – *The function $k(\phi) = \sqrt{\frac{2}{\pi}} \ln\left(\coth \frac{\pi|\phi|}{4}\right)$ has the following properties:*

1. $k(\phi) = \dfrac{2\sqrt{2}}{\sqrt{\pi}} \displaystyle\sum_{n=0}^{\infty} \frac{e^{-\frac{2n+1}{2}\pi|\phi|}}{2n+1}$ *for any* $|\phi| > 0$ *and*
2. $k(\phi)e^{\delta|\phi|} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ *for any* δ *with* $0 \leq \delta < \dfrac{\pi}{2}$.

Proof. – The Taylor series expansion

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

is valid for any $x \in (-1, 1)$. Take $x = e^{-\pi|\phi|/2}$ to obtain the first property. The second property is a direct consequence of the first. □

THEOREM 4.2.2. – *Suppose that the measurable function $\omega(\phi)$ is a solution of Equation* (4.0.1) *and that it satisfies the conditions:*

$$\omega(\phi) = -\omega(-\phi) \quad \textit{for almost every } \phi \in \mathbb{R} \textit{ and } \quad 0 \leq \omega(\phi) \leq \frac{\pi}{2} \quad \textit{for almost every } \phi \geq 0.$$

Then $\sin \omega(\phi)$, $\omega(\phi)$ *and* $F_\gamma \omega(\phi) \in L_1 \cap L_2$.

Proof. – Assume that $\sin\omega(\phi) \notin L_1$. Then for any fixed number ϵ satisfying $0 < \epsilon < 1/2\gamma\tilde{c}$, there exists an $A > 0$ such that

$$\int_0^M \sin\omega(\tau)d\tau > \frac{1}{3\epsilon\gamma} \tag{4.2.1}$$

for any $M \geq A$, where $\tilde{c} = \int_0^\infty k(t)dt + 1/2$.

Since $k(\phi)$ is even, Equation (4.0.1) can be expressed as

$$\begin{aligned}\omega(\phi) =& \frac{1}{\sqrt{2\pi}}\int_0^\infty k(t)\frac{\gamma\sin\omega(\phi-t)}{1+3\gamma\int_0^{\phi-t}\sin\omega(\tau)d\tau}dt \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^\infty k(t)\frac{\gamma\sin\omega(\phi+t)}{1+3\gamma\int_0^{\phi+t}\sin\omega(\tau)d\tau}dt = I + II,\end{aligned}$$

say. Consider the integral

$$\int_A^\infty IIe^{-\delta\phi}d\phi = \frac{1}{\sqrt{2\pi}}\int_0^\infty k(t)dt\int_A^\infty \frac{\gamma\sin\omega(\phi+t)e^{-\delta\phi}}{1+3\gamma\int_0^{\phi+t}\sin\omega(\tau)d\tau}d\phi,$$

where δ is any constant in the range $0 < \delta < \frac{\pi}{2}$ for which $\int_0^\infty k(t)\cosh(\delta t)\,dt < \tilde{c}$. It follows from (4.2.1) and the hypotheses on ω that:

$$\begin{aligned}\int_A^\infty IIe^{-\delta\phi}d\phi &\leq \epsilon\gamma\int_0^\infty k(t)dt\int_A^\infty \omega(\phi+t)e^{-\delta\phi}d\phi \\ &= \epsilon\gamma\int_0^\infty k(t)dt\int_{A+t}^\infty \omega(\tau)e^{-\delta(\tau-t)}d\tau \\ &\leq \epsilon\gamma\int_0^\infty k(t)e^{\delta t}dt\int_A^\infty \omega(\tau)e^{-\delta\tau}d\tau.\end{aligned} \tag{4.2.2}$$

Now consider the integral

$$\int_A^\infty Ie^{-\delta\phi}d\phi = \frac{1}{\sqrt{2\pi}}\int_0^\infty k(t)dt\int_A^\infty \frac{\gamma\sin\omega(\phi-t)e^{-\delta\phi}}{1+3\gamma\int_0^{\phi-t}\sin\omega(\tau)d\tau}d\phi.$$

Let $\eta = \phi - t$ and use (4.2.1) to obtain

$$\begin{aligned}\int_A^\infty Ie^{-\delta\phi}d\phi &= \frac{1}{\sqrt{2\pi}}\int_0^\infty k(t)e^{-\delta t}dt\int_{A-t}^\infty \frac{\gamma\sin\omega(\eta)e^{-\delta\eta}}{1+3\gamma\int_0^\eta\sin\omega(\tau)d\tau}d\eta \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty k(t)e^{-\delta t}dt\left(\int_A^\infty + \int_{A-t}^A\right)\frac{\gamma\sin\omega(\eta)e^{-\delta\eta}}{1+3\gamma\int_0^\eta\sin\omega(\tau)d\tau}d\eta \\ &\leq \epsilon\gamma\int_0^\infty k(t)e^{-\delta t}dt\int_A^\infty \omega(\eta)e^{-\delta\eta}d\eta + \gamma\int_0^\infty tk(t)dt.\end{aligned} \tag{4.2.3}$$

It follows from (4.0.1), (4.2.2), (4.2.3) and the inequality $\int_0^\infty k(t)\cosh(\delta t)\,dt < \tilde{c}$ that

$$\begin{aligned}\int_A^\infty \omega(\phi)e^{-\delta\phi}d\phi &= \int_A^\infty Ie^{-\delta\phi}d\phi + \int_A^\infty IIe^{-\delta\phi}d\phi \\ &\le 2\epsilon\gamma\int_0^\infty k(t)\cosh\delta t dt\int_A^\infty \omega(\phi)e^{-\delta\phi}d\phi + \gamma\int_0^\infty tk(t)dt \\ &< 2\epsilon\gamma\tilde{c}\int_A^\infty \omega(\phi)e^{-\delta\phi}d\phi + \gamma\int_0^\infty tk(t)dt,\end{aligned}$$

whence

$$\int_A^\infty \omega(\phi)e^{-\delta\phi}d\phi \le \frac{\gamma}{1-2\epsilon\gamma\tilde{c}}\int_0^\infty tk(t)dt.$$

Applying Fatou's lemma yields

$$\begin{aligned}\int_A^\infty \omega(\phi)d\phi &= \int_A^\infty \liminf_{\delta\to 0}\omega(\phi)e^{-\delta\phi}d\phi \\ &\le \liminf_{\delta\to 0}\int_A^\infty \omega(\phi)e^{-\delta\phi}d\phi \le \frac{\gamma}{1-2\epsilon\gamma\tilde{c}}\int_0^\infty tk(t)dt < \infty.\end{aligned}$$

The hypotheses on ω and the above inequality imply $\omega \in L_1(\mathbb{R})$.

However, since $|\sin\omega| \le |\omega|$, it follows that $\sin\omega \in L_1(\mathbb{R})$, a contradiction. Hence the assumption is false and $\sin\omega \in L_1(\mathbb{R})$.

Because both $k(\phi)$ and $\sin\omega(\phi)$ belong to L_1, Young's inequality implies:

$$\|\omega\|_1 \le \|k * F_\gamma\omega\|_1 \le \gamma\|k * |\sin\omega|\|_1 \le \gamma\|k\|_1\|\sin\omega\|_1.$$

This means that $\omega \in L_1(\mathbb{R})$. Because ω satisfies the convolution equation (4.0.1), it is plainly continuous. Consequently, both $\sin\omega$ and ω belong to $L_2(\mathbb{R})$. Using the inequality $|F_\gamma\omega(\phi)| \le |\gamma\sin\omega(\phi)|$ then yields $F_\gamma\omega \in L_1 \cap L_2$. □

A consequence of Theorem 4.2.2 is that ω, $\sin\omega$ and $F_\gamma\omega$ are all H^∞-functions, as was pointed out in the work of Amick and Toland [3], and Benjamin *et al.* [5]. Appropriate estimation of norms of $\sin\omega$ and $F_\gamma\omega$ will allow us to conclude the three functions ω, $\sin\omega$ and $F_\gamma\omega$ have analytic extensions to a strip in the complex plane. Since the verification of the relevant estimates closely follows the style of the proof of Theorem 4.1.4, it will only be outlined.

THEOREM 4.2.3. – *Suppose that ω is a solution of Equation* (4.0.1) *satisfying the conditions in Theorem 4.2.2. Then ω, $\sin\omega$ and $F_\gamma\omega$ all lie in $H^\infty(\mathbb{R})$. Furthermore, there exists a constant $\sigma > 0$, such that all three functions have analytic extensions to the strip*

$$S_\sigma = \{w = \phi + i\psi \in \mathbb{C} : \ |\psi| < \sigma\}.$$

Proof. – Using the fact that the Fourier transform $\hat{k}$ of k in Equation (4.0.1) satisfies the inequality $|\xi\hat{k}(\xi)| = |\tanh\xi| \leq 1$, one obtains formally the estimate

$$\left|\left(\omega^{(m)},\, f'\right)\right| = \int_{-\infty}^{\infty} \hat{k}(\xi)\widehat{(F_\gamma\omega)}^{(m)}(\xi)i\xi\overline{\hat{f}(\xi)}d\xi \leq \int_{-\infty}^{\infty}\left|\hat{f}(\xi)\right|\left|\widehat{(F_\gamma\omega)}^{(m)}(\xi)\right|d\xi$$
$$\leq \|\hat{f}\|_2\|\widehat{(F_\gamma\omega)}^{(m)}\|_2 = \|f\|_2\|(F_\gamma\omega)^{(m)}\|_2,$$

for any $f \in C_c^\infty$ and any integer $m \geq 1$. This implies immediately that

$$\left\|\omega^{(m)}\right\|_2 \leq \left\|(F_\gamma\omega)^{(m-1)}\right\|_2 \tag{4.2.4}$$

for all $m \geq 1$. Note also that formally,

$$(\sin\omega)^{(n)} = \frac{d^n\sin\omega}{d\phi^n} = \omega^{(n)}\cos\omega + (\omega')^n\sin\left(\omega + \frac{n\pi}{2}\right)$$
$$+\sum_{s=2}^{n-1}\frac{\sin(\omega+\frac{s\pi}{2})}{s!}\sum^{(s,n)}\binom{n}{j_1,\cdots,j_{s-1}}\omega^{(n-j_1)}\omega^{(j_1-j_2)}\cdots\omega^{(j_{s-1})}, \tag{4.2.5}$$

and

$$(F_\gamma\omega)^{(n)} = \frac{\gamma(\sin\omega)^{(n)} - 3\gamma\sum_{j=1}^n\binom{n}{j}(\sin\omega)^{(j-1)}(F_\gamma\omega)^{(n-j)}}{1+3\gamma\int_0^\phi\sin\omega(t)dt} \tag{4.2.6}$$

for any integer $n \geq 1$. Since it is known from Theorem 4.2.2 that $F_\gamma\omega \in L_2$, it follows from (4.2.4) that $\omega \in H^1$. It then follows from (4.2.5) that $\sin\omega \in H^1$, and afterward from (4.2.6) that $F_\gamma\omega \in H^1$. Continuing this argument inductively leads to the conclusion that ω, $\sin\omega$ and $F_\gamma\omega$ all lie in H^∞.

Define the two positive quantities:

$$a_1 = \max_{n=1,2}\{\|(\sin\omega)^{(n)}\|_2^{\frac{1}{2n-1}},\ \|(F_\gamma\omega)^{(n)}\|_2^{\frac{1}{2n}},\ \|(\sin\omega)^{(n)}\|_\infty^{\frac{1}{2n+1}}\},$$

and

$$a = \max\left\{a_1,\ \frac{1}{3} + (1+\|F_\gamma\omega\|_2)(e^{\|k\|_2}-1), \gamma(7+\|F_\gamma\omega\|_2),\ \|k\|_2e^{\|k\|_2}\right\}.$$

Then the inequalities

$$\begin{cases} \|(\sin\omega)^{(n)}\|_2 \leq a^{2n-1}n^{n-1}, \\ \|(F_\gamma\omega)^{(n)}\|_2 \leq a^{2n}n^{n-1}, \\ \|(\sin\omega)^{(n)}\|_\infty \leq a^{2n+1}n^{n-1}, \end{cases} \tag{4.2.7}$$

hold for $n = 1, 2$. Arguing by induction in a way similar to that used in evaluating $\|(G(f))^{(n)}\|_2$ in the proof of Theorem 4.1.4, and making use of the relation

$$\omega^{(m)} = k * (F_\gamma \omega)^{(m)} \tag{4.2.8}$$

for $m \geq 0$ together with (4.2.4), Lemma 4.1.2 and the expressions (4.2.5) and (4.2.6) for $(\sin \omega)^{(n)}$ and $(F_\gamma \omega)^{(n)}$, respectively, one determines that (4.2.7) holds for all integers $n \geq 1$.

The inequalities in (4.2.7) together with (4.2.8) show that the functions ω, $\sin \omega$ and $F_\gamma \omega$ all have Taylor series expansions about any point $\phi \in \mathbb{R}$ with radius of convergence not less than $1/a^2 e$. This implies the desired conclusion. □

An immediate consequence of Theorem 4.2.3 is the exponential decay property of the Fourier transforms of the functions ω, $\sin \omega$ and $F_\gamma \omega$.

COROLLARY 4.2.4. – *The Fourier transforms $\hat{\omega}$, $\widehat{F_\gamma \omega}$ and $\widehat{\sin \omega}$ of ω, $F_\gamma \omega$ and $\sin \omega$ satisfy the inequalities*

$$\int_{-\infty}^{\infty} |\hat{\omega}(t)|^2 e^{2\mu|t|} dt < \infty,$$

$$\int_{-\infty}^{\infty} \left|\widehat{F_\gamma \omega}(t)\right|^2 e^{2\mu|t|} dt < \infty,$$

$$\int_{-\infty}^{\infty} \left|\widehat{\sin \omega}(t)\right|^2 e^{2\mu|t|} dt < \infty,$$

respectively, for any μ with $0 < \mu < \frac{1}{a^2 e}$.

Proof. – This follows from the Paley-Wiener theory as in the proof of Corollary 4.1.5. □

Thus far, we have discussed analyticity of the solution $\omega(\phi)$ to Equation (4.0.1). Notice that $\omega(\phi)$ is actually the boundary value of the harmonic function $\omega(\phi, \psi)$ at the top of the region $\{(\phi, \psi) : -\infty < \phi < \infty, 0 < \psi < 1\}$ of its definition. The value $-\omega(\phi, \psi)$ represents the angle between the streamline indexed by the value ψ and the positive real axis at the correspondingly transformed point (ϕ, ψ) in the flow region. In addition, $\omega(\phi, \psi)$ can be expressed as the integral

$$\omega(\phi, \psi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(\phi - t, \psi) F_\gamma \omega(t) dt, \tag{4.2.9}$$

where

$$k(\phi, \psi) = \frac{1}{\sqrt{2\pi}} \ln \frac{\cosh \frac{\pi|\phi|}{2} + \sin \frac{\pi\psi}{2}}{\cosh \frac{\pi|\phi|}{2} - \sin \frac{\pi\psi}{2}},$$

whose Fourier symbol is $\hat{k}(\xi, \psi) = \frac{1}{\xi} \frac{\sinh \psi\xi}{\cosh \xi}$. Thus it is natural to speculate that $\omega(\phi, \psi)$ possesses analyticity properties similar to those of its boundary value $\omega(\phi)$. As a matter of fact, one conclusion that can be easily drawn by applying the argument in Theorem 4.2.3 to $\omega(\phi, \psi)$ is that it has a continuous extension to the infinite rectangular cylinder

$$\{(w_1, \psi) : -\infty < \Re w_1 < \infty, |\Im w_1| < \sigma, 0 < \psi \leq 1\}$$

in such a way that the extension is a holomorphic function with respect to w_1 and it is a C^∞-function with respect to both w_1 and ψ. In the next section, it will be shown that $\omega(\phi, \psi)$ and its complex conjugate can in fact be extended to functions holomorphic with respect to two complex variables.

4.3. Analytic extension of the function $\omega(\phi, \psi)$

To show analyticity of $\omega(\phi, \psi)$ and its complex conjugate $\ln q(\phi, \psi)$ in the next theorem, advantage is taken of both analyticity and the exponential decay property of the function $F_\gamma\omega(\phi)$ which is the derivative with respect to ϕ of the boundary value at $\psi = 1$ of the function $\ln q(\phi, \psi)$, where $q(\phi, \psi)$ is the speed of the flow normalized so that $q(\phi, \psi) \to 1$ as $|\phi| \to \infty$ (see [5]).

THEOREM 4.3.1. – *Suppose that $\omega(\phi)$ is a solution of Equation* (4.0.1), *satisfying the conditions in Theorem* 4.2.2. *Then the corresponding harmonic function $\omega(\phi, \psi)$ has an extension as a holomorphic function $\omega(w_1, w_2)$ defined on the open set*

$$D_\rho = \{(w_1, w_2) : |\Im w_1| + |\Re w_2| < 1 + \rho\},$$

in $\mathbb{C}^2$, where the constant ρ is defined by

$$\rho = \max\left\{\sigma > 0; \int_{-\infty}^{\infty} \left|\widehat{F_\gamma\omega}(\xi)\right|^2 e^{2\sigma|\xi|} d\xi < \infty\right\}. \tag{4.3.1}$$

Proof. – For any constant ν with $0 < \nu < \rho$, let $D_\nu = \{(w_1, w_2) \in \mathbb{C}^2 : |\Im w_1| + |\Re w_2| < 1 + \nu\}$. We shall show that the integral

$$\omega(w_1, w_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{k}(\xi, w_2)\widehat{F_\gamma\omega}(\xi)e^{-iw_1\xi} d\xi \tag{4.3.2}$$

defines a holomorphic function on the open set D_ν. For any point $(w_1, w_2) = (\phi + i\tau, \psi + i\eta) \in D_\nu$, the following estimate is valid:

$$\begin{aligned} &|\hat{k}(\xi, w_2)\widehat{F_\gamma\omega}(\xi)e^{|\tau\xi|}| \\ &= \left|\frac{\cos\eta\xi \sinh\psi\xi + \mathrm{i}\sin\eta\xi\cosh\psi\xi}{\xi\cosh\xi}\widehat{F_\gamma\omega}(\xi)e^{|\tau\xi|}\right| \\ &\leq 3e^{(|\psi|-1+|\tau|)|\xi|}\left|\frac{\widehat{F_\gamma\omega}(\xi)}{\xi}\right| = 3e^{(|\Im w_1|-1+|\Re w_2|)|\xi|}\left|\frac{\widehat{F_\gamma\omega}(\xi)}{\xi}\right|. \end{aligned} \tag{4.3.3}$$

Because $\widehat{F_\gamma\omega}$ is an analytic function by Corollary 3.2.5 and $\widehat{F_\gamma\omega}(0) = 0$, there is an $M > 0$, such that

$$\left|\frac{\widehat{F_\gamma\omega}(\xi)}{\xi}\right| \leq M \tag{4.3.4}$$

for $|\xi| < 1$. It follows from (4.3.1), (4.3.3), (4.3.4) and the definition of D_ν that

$$\int_{-\infty}^{\infty} |\hat{k}(\xi, w_2)\widehat{F_\gamma \omega}(\xi)| e^{|\tau\xi|} d\xi \le 2M \int_{-1}^{1} e^{|\tau\xi|} d\xi + \\ + 3\int_{|\xi|>1} |\widehat{F_\gamma \omega}(\xi)| e^{(|\psi|-1+|\tau|)|\xi|} d\xi \\ \le 2M \int_{-1}^{1} e^{(1+\nu)|\xi|} d\xi + 3\int_{|\xi|>1} |\widehat{F_\gamma \omega}(\xi)| e^{\nu|\xi|} d\xi < \infty.$$

Therefore, $\omega(w_1, w_2)$ is a well defined and bounded function on D_ν. For any point $(w_1, w_2) \in D_\nu$, there is a $\delta > 0$ such that if $|\Delta w_2| \le \delta$, then $|\Re w_2| + |\Re \Delta w_2| + |\Im w_1| - 1 \le |\Re w_2| + \delta + |\Im w_1| - 1 < \nu$. For such a δ and such values of Δw_2, the inequality

$$\left|\frac{\hat{k}(\xi, w_2 + \Delta w_2) - \hat{k}(\xi, w_2)}{\Delta w_2}\right| = \left|\int_0^1 \frac{\partial \hat{k}}{\partial w_2}(\xi, w_2 + \theta\Delta w_2) d\theta\right| \\ = \left|\int_0^1 \frac{\cosh(\mathrm{w}_2 + \theta \Delta \mathrm{w}_2)\xi}{\cosh \xi} d\theta\right| \le e^{(|\Re w_2| + |\Re \Delta w_2| - 1)|\xi|} \tag{4.3.5}$$

implies that

$$\left|\frac{\hat{k}(\xi, w_2 + \Delta w_2) - \hat{k}(\xi, w_2)}{\Delta w_2} \widehat{F_\gamma \omega}(\xi) e^{-iw_1\xi}\right| \\ \le e^{(|\Re w_2| + \delta + |\Im w_1| - 1)|\xi|} |\widehat{F_\gamma \omega}(\xi)|. \tag{4.3.6}$$

Because $\nu < \rho$, it follows from (4.3.1) that the right-hand side of (4.3.6) is integrable over $\mathbb{R}$ with respect to ξ. An application of the Dominated-Convergence Theorem then yields

$$\lim_{\Delta w_2 \to 0} \int_{-\infty}^{\infty} \frac{\hat{k}(\xi, w_2 + \Delta w_2) - \hat{k}(\xi, w_2)}{\Delta w_2} \widehat{F_\gamma \omega}(\xi) e^{-iw_1\xi} d\xi \\ = \int_{-\infty}^{\infty} \frac{\partial \hat{k}}{\partial w_2}(\xi, w_2) \widehat{F_\gamma \omega}(\xi) e^{-iw_1\xi} d\xi.$$

The last computation shows simultaneously that $\omega(w_1, w_2)$ is complex differentiable with respect to w_2 and that its partial derivative $\frac{\partial \omega}{\partial w_2}$ is bounded on D_ν. Similarly, using the estimate

$$\left|\xi \hat{k}(\xi, w_2) \widehat{F_\gamma \omega}(\xi) \frac{e^{-iw_1\xi}(e^{-i\Delta w_1 \xi} - 1)}{i\Delta w_1 \xi}\right| \\ \le 2e^{(|\Im w_1| + |\Delta \Im w_1| + |\Re w_2| - 1)|\xi|} \left|\widehat{F_\gamma \omega}(\xi)\right| \tag{4.3.7}$$

and the Dominated-Convergence Theorem again shows that $\omega(w_1, w_2)$ is also complex differentiable with respect to w_1 with $\frac{\partial \omega}{\partial w_1}$ bounded on D_ν.

Thus, $\omega(w_1, w_2)$ is an analytic function on D_ν. Because D_ρ is the union of such open sets D_ν, it follows that $\omega(w_1, w_2)$ is a holomorphic function of w_1 and w_2 on D_ρ. Moreover, (4.2.9) and (4.3.2) show that $\omega(w_1, w_2)$ is an extension of $\omega(\phi, \psi)$. □

Now consider the complex conjugate $\ln q(\phi, \psi)$ of $\omega(\phi, \psi)$, expressed in the form of the inverse Fourier transform

$$\ln q(\phi, \psi) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cosh \psi\xi}{\xi \cosh \xi} \widehat{F_\gamma \omega}(\xi) e^{-i\phi\xi} d\xi. \tag{4.3.8}$$

That $\ln q$ is indeed the conjugate of ω follows upon checking that the right-hand side of (4.3.8), call it g, and $\omega(\phi, \psi)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial g}{\partial \phi} = \frac{\partial \omega}{\partial \psi}, \quad \frac{\partial g}{\partial \psi} = -\frac{\partial \omega}{\partial \phi}.$$

This means that $g(\phi, \psi) = \ln q(\phi, \psi) + \text{const}$. Since $\lim_{|\phi| \to \infty} \ln q(\phi, \psi) = \lim_{|\phi| \to \infty} g(\phi, \psi) = 0$, $g(\phi, \psi) = \ln q(\phi, \psi)$.

Like $\omega(\phi, \psi)$, $\ln q(\phi, \psi)$ also has an analytic extension to a holomorphic function, denoted by $\ln q(w_1, w_2)$, of two complex variables in D_ρ, a fact which may be proved by an argument similar to that used in the proof of Theorem 4.3.1.

THEOREM 4.3.2. – *Suppose that $\omega(\phi)$ is a solution of Equation* (4.0.1), *satisfying the conditions in Theorem* 4.2.2. *Then, the complex conjugate $\ln q(\phi, \psi)$ of the harmonic function $\omega(\phi, \psi)$ can be extended as a holomorphic function expressed by the integral*

$$\ln q(w_1, w_2) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cosh w_2\xi}{\xi \cosh \xi} \widehat{F_\gamma \omega}(\xi) e^{-iw_1\xi} d\xi \tag{4.3.9}$$

defined on the open set D_ρ.

Remarks. – It is worth noting that the two holomorphic functions $\omega(w_1, w_2)$ and $\ln q(w_1, w_2)$ preserve complex versions of the Cauchy-Riemann equations, viz.

$$\begin{aligned} \frac{\partial \ln q}{\partial w_1} &= \frac{\partial \omega}{\partial w_2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cosh w_1\xi}{\cosh \xi} \widehat{F_\gamma \omega}(\xi) e^{-iw_1\xi} d\xi, \\ \frac{\partial \ln q}{\partial w_2} &= -\frac{\partial \omega}{\partial w_1} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh w_2\xi}{\cosh \xi} \widehat{F_\gamma \omega}(\xi) e^{-iw_1\xi} d\xi. \end{aligned} \tag{4.3.10}$$

In addition, by studying the integrals in (4.3.2) and (4.3.8), one may also obtain exponential decay properties of the two harmonic functions $\omega(\phi, \psi)$ and $\ln q(\phi, \psi)$. Take the function $\omega(\phi, \psi)$ for example. For any $\psi \in [0, 1]$, it follows from Corollary 3.2.5 that $\frac{\sinh \psi\xi}{\xi \cosh \xi} \widehat{F_\gamma \omega}(\xi)$ is a holomorphic function of ξ defined on the strip $\{\xi \in \mathbb{C} : |\Im\xi| < \eta_1\}$, and

$$\sup_{0 < \mu < \mu_0} \int_{-\infty}^{\infty} \left| \frac{\sinh \psi(\xi + i\mu)}{(\xi + i\mu) \cosh (\xi + i\mu)} \widehat{F_\gamma \omega}(\xi + i\mu) \right|^2 d\xi < \infty,$$

for any μ_0 with $0 < \mu_0 < \eta_1$. The Paley-Wiener theorem implies that

$$\int_{-\infty}^{\infty} |\omega(\phi,\psi)|^2 e^{2\sigma|\phi|} d\phi < \infty$$

for any σ with $0 < \sigma < \eta_1$. In consequence, the convolution equation (4.2.9) implies $\omega(\phi,\psi)e^{\sigma|\phi|} \in L_\infty$ for any fixed $\psi \in [-1,1]$.

Recall that if $w = f(z)$ is the conformal mapping of the fluid region $\overline{\Omega}$ to the strip $\overline{D} = \{(\phi,\psi) : -\infty < \phi < \infty, 0 \le \psi \le 1\}$, then

$$\begin{aligned} \frac{dz}{dw} &= \frac{e^{-i\omega(\phi,\psi)}}{cq(\phi,\psi)} = \frac{\cos\omega}{cq} - i\frac{\sin\omega}{cq} \\ &= \frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = \frac{\partial y}{\partial \psi} - i\frac{\partial x}{\partial \psi}. \end{aligned} \tag{4.3.11}$$

Therefore, an analytic extension of the inverse function $z = f^{-1}(w)$ of $w = f(z)$ can be inferred from that of ω and $\ln q$.

THEOREM 4.3.3. – *The function $z = x(\phi,\psi) + iy(\phi,\psi)$ defined on $\overline{D}$ can be extended to a holomorphic function of two complex variables in the open set D_ρ.*

Proof. – Let $g_1(w_1,w_2) = \frac{\cos\omega(w_1,w_2)}{cq(w_1,w_2)}$ and $g_2(w_1,w_2) = \frac{\sin\omega(w_1,w_2)}{cq(w_1,w_2)}$, where $q(w_1,w_2)$ is defined as $q(w_1,w_2) = e^{\ln q(w_1,w_2)}$. Since $\frac{\partial g_1}{\partial w_2} = -\frac{\sin\omega}{cq}\omega_{w_2} - \frac{\cos\omega}{cq^2}q_{w_2}$, and $\frac{\partial g_2}{\partial w_1} = \frac{\cos\omega}{cq}\omega_{w_1} - \frac{\sin\omega}{cq^2}q_{w_1}$, it follows from the definition of q and (4.3.10) that

$$q_{w_2} = q(\ln q)_{w_2} = -q\omega_{w_1}, \quad \text{and} \quad q_{w_1} = q(\ln q)_{w_1} = q\omega_{w_2}$$

and thus, $\frac{\partial g_1}{\partial w_2} = -\frac{\sin\omega}{cq}\omega_{w_2} + \frac{\cos\omega}{cq}\omega_{w_1} = \frac{\partial g_2}{\partial w_1}$. Because D_ρ is a simply-connected domain, it is adduced from Cauchy's theorem [18] that there is a holomorphic function $X(w_1,w_2)$, uniquely determined up to an additive constant and defined on D_ρ, such that $\frac{\partial X}{\partial w_1} = g_1$ and $\frac{\partial X}{\partial w_2} = g_2$. Similarly, the relation $\frac{\partial g_1}{\partial w_1} = \frac{\partial(-g_2)}{\partial w_2}$, which is also verified by using (4.3.10), implies the existence (see [18] again) of a holomorphic function $Y(w_1,w_2)$, unique up to some additive constant, defined on D_ρ and such that $\frac{\partial Y}{\partial w_1} = -g_2$ and $\frac{\partial Y}{\partial w_2} = g_1$. Because of (4.3.11), we see that g_1 and g_2 are analytic extensions of the partial derivatives of $x(\phi,\psi)$ and $y(\phi,\psi)$. It follows that $X(w_1,w_2)$ and $Y(w_1,w_2)$ can be chosen in such a way that they are analytic extensions of x and y. □

To obtain an analytic extension of the conformal mapping $w = f(z)$, we shall apply the Open-Mapping Theorem. The use of the open-mapping result requires some estimates for related functions. The next theorem deals with this issue.

THEOREM 4.3.4. – *Provided that $\alpha = \max_{\phi\in\mathbf{R}} |\omega(\phi)| < \pi/2$, there exists a constant $\mu > 0$ for which the functions*

$$\frac{1}{\cos\omega(w_1,w_2)}, \qquad \tan\omega(w_1,w_2), \qquad e^{i\omega(w_1,w_2)},$$

the first-order derivatives of ω and the second-order derivatives of the functions X and Y are all bounded holomorphic functions on the open set:

$$S_\mu = \{(w_1, w_2) \in \mathbb{C} : |\Im w_1| < \mu,\ |\Im w_2| < \mu,\ |\Re w_2| < 1 + \mu\}.$$

Proof. – It follows from the proof of Theorem 4.3.1 that ω, $\ln q$ and their partial derivatives are all bounded holomorphic functions on the set $D_\nu = \{(w_1, w_2) \in \mathbb{C}^2 : |\Im w_1| + |\Re w_2| < 1 + \nu\}$ for any ν with $0 < \nu < \rho$. Then the functions $e^{i\omega(w_1, w_2)}$, q and the second-order derivatives of X and Y, expressed in terms of $\sin\omega$, $\cos\omega$, q and first order derivatives of ω, are also bounded functions on D_ν. Fix a $\nu \in (0, \rho)$, and let M be a simultaneous upper bound for all of these functions on D_ν. To finish the proof, we show that there is a μ with $0 < \mu \le \nu/2$ such that $|\cos\omega(w_1, w_2)|$ has a positive lower bound on S_μ.

Let ϵ be a positive number such that when $|z| \le \epsilon$, $|\sin z| + 2|\sin^2 z/2| < (1 - \sin\alpha)/2$, where $\alpha < \frac{\pi}{2}$ by hypothesis. It follows from the definition of M that the inequality

$$|\Delta\omega| = |\omega(w_1, w_2) - \omega(\phi, \psi)| \le M|w_1 - \phi| + M|w_2 - \psi| < \epsilon \tag{4.3.12}$$

is valid if $\phi = \Re w_1$, $|\Im w_1| < \min\{\nu/2, \epsilon/2M\}$, $|\Re(w_2 - \psi)| < \min\{\nu/2, \epsilon/4M\}$ and $|\Im w_2| < \epsilon/4M$, where $\psi \in [-1, 1]$ and $\Delta\omega = \omega(w_1, w_2) - \omega(\phi, \psi)$. Having so chosen ϵ, one then infers the existence of a positive constant $\mu \le \nu/2$ such that (4.3.12) holds for any $(w_1, w_2) \in S_\mu$ by choosing $\phi = \Re w_1$ and $\psi \in [-1, 1]$ satisfying the identity $|\Re w_2 - \psi| = \inf\limits_{\eta \in [-1,1]} |\Re w_2 - \eta|$. Then it transpires that:

$$\begin{aligned} |\sin\omega(w_1, w_2) - \sin\omega(\phi, \psi)| &= |\sin\Delta\omega\cos\omega(\phi, \psi) + (\cos\Delta\omega - 1)\sin\omega(\phi, \psi)| \\ &\le |\sin\Delta\omega| + 2|\sin^2\frac{\Delta\omega}{2}| < (1 - \sin\alpha)/2. \end{aligned} \tag{4.3.13}$$

Because $\omega(\phi, \psi)$ is a harmonic function on the closure $\overline{D_1}$ of the domain

$$D_1 = \{(\phi, \psi) : -\infty < \phi < \infty, -1 < \psi < 1\}$$

and since $\alpha = \max\limits_{\phi \in \mathbf{R}} |\omega(\phi)| = \max\limits_{\phi \in \mathbf{R}} |\omega(\phi, -1)|$ and $\lim\limits_{|\phi| \to \infty} \omega(\phi, \psi) = 0$ for any $\psi \in [-1, 1]$, it follows from the Maximum Principle that the inequality $|\omega(\phi, \psi)| \le \alpha$ holds on $\overline{D_1}$. This result and (4.3.13) lead to the inequality $|\sin\omega(w_1, w_2)| \le (1 + \sin\alpha)/2$ for any $(w_1, w_2) \in S_\mu$. Then $|\cos^2\omega(w_1, w_2)| \ge 1 - |\sin^2\omega(w_1, w_2)| \ge 1 - (1 + \sin\alpha)^2/4 > 0$ holds for any $(w_1, w_2) \in S_\mu$. □

With the above preparation, we begin investigating the analyticity of the velocity potential and the stream function by studying the holomorphic mapping $\begin{pmatrix} X \\ Y \end{pmatrix} : D_\rho \to \mathbb{C}^2$. Because the Jacobian of the mapping $\begin{pmatrix} X \\ Y \end{pmatrix}$ is

$$\det\begin{vmatrix} X_{w_1} & X_{w_2} \\ Y_{w_1} & Y_{w_2} \end{vmatrix} = \det\begin{vmatrix} \frac{\cos\omega}{cq} & \frac{\sin\omega}{cq} \\ -\frac{\sin\omega}{cq} & \frac{\cos\omega}{cq} \end{vmatrix} = \frac{1}{c^2q^2} \ne 0$$

on D_ρ, $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a locally homeomorphic mapping, and therefore it maps D_ρ to an open subset of $\mathbb{C}^2$. Our results in the next subsection will follow from this property and the fact that $\begin{pmatrix} X \\ Y \end{pmatrix}\Big|_{\overline{D_1}}$ is a conformal mapping of $\overline{D_1}$ onto the closed set

$$\overline{\Omega_1} = \{(x, y) : -\infty < x < \infty, -H(x) \le y \le H(x)\}.$$

4.4. The velocity potential ϕ and the stream function ψ

As mentioned already, Benjamin *et al.* [5] proved the existence of a solitary-wave solution ω of (4.0.1) such that:

$$(4.4.1)\quad \omega(\phi) = -\omega(-\phi) \text{ for any } \phi \in \mathbb{R}, \text{ and } 0 \le \omega(\phi) \le \phi/3 \text{ for any } \phi \in [0, \infty).$$

Using these properties, we shall prove that the corresponding holomorphic mapping $\begin{pmatrix} X \\ Y \end{pmatrix}\Big|_{\overline{D_1}}$ is a homeomorphism of $\overline{D_1}$ onto $\overline{\Omega_1}$.

THEOREM 4.4.1. – *Let ω be a solution of* (4.0.1) *satisfying the conditions of* (4.4.1). *Then the holomorphic mapping $\begin{pmatrix} X \\ Y \end{pmatrix}$ corresponding to ω is a homeomorphism of $\overline{D_1}$ onto $\overline{\Omega_1}$.*

Proof. – It follows from Theorem 4.3.1, Theorem 4.3.2 and the maximum principle that ω and $\ln q$ can be extended to harmonic functions which are complex conjugate to each other on $\overline{D_1}$ with the symmetry properties

$$(4.4.2)\quad \begin{aligned} \ln q(\phi, \psi) &= \ln q(-\phi, \psi), \quad \ln q(\phi, \psi) = \ln q(\phi, -\psi), \\ \omega(\phi, \psi) &= -\omega(-\phi, \psi), \quad \omega(\phi, \psi) = -\omega(\phi, -\psi), \end{aligned}$$

for any $(\phi, \psi) \in \overline{D_1}$ and, moreover, $\max_{(\phi,\psi)\in\overline{D_1}} |\omega(\phi, \psi)| \le \frac{\pi}{3}$.

Choose any constant $N > 0$, construct the closed path l in $\overline{D_1}$ as

$$l = \{|\phi| \le N, \psi = 1\} \cup \{\phi = -N, |\psi| \le 1\} \cup \{|\phi| \le N, \psi = -1\} \cup \{\phi = N, |\psi| \le 1\}$$

and let D_l be the interior of l. Fix a point of the form $(-N, \psi_0) \in l$. Then for any point $(N, \psi) \in l$, consider

$$\begin{pmatrix} \Delta X \\ \Delta Y \end{pmatrix} = \begin{pmatrix} X(N, \psi) - X(-N, \psi_0) \\ Y(N, \psi) - Y(-N, \psi_0) \end{pmatrix} = \begin{pmatrix} \int_{-N}^{N} \frac{\cos\omega(s,\psi)}{cq(s,\psi)}\, ds + \int_{\psi_0}^{\psi} \frac{\sin\omega(-N,t)}{cq(-N,t)}\, dt \\ \int_{\psi_0}^{\psi} \frac{\cos\omega(-N,t)}{cq(-N,t)}\, dt \end{pmatrix}.$$

If $\psi \neq \psi_0$, $\Delta Y \neq 0$ since $\cos\omega(\phi, \psi) > 0$ on $\overline{D_1}$. If $\psi = \psi_0$, $\Delta X \neq 0$, for the same reason.

In a similar way, one may show by using (4.4.2) that $\begin{pmatrix} X \\ Y \end{pmatrix}$ is an one-to-one mapping of l onto a closed path Γ. Then it is a conformal mapping of D_l onto the interior of Γ. Since N is arbitrary, this leads to the desired conclusion. □

In Theorem 4.4.2, use will be made of the norm $|\vec{z}| = \sqrt{|z_1|^2 + |z_2|^2}$ for any $\vec{z} = (z_1, z_2) \in \mathbb{C}^2$. Denote the ball of radius r with the center (z_{10}, z_{20}) by

$B((z_{10}, z_{20}), r) = \{(z_1, z_2) : |(z_1, z_2) - (z_{10}, z_{20})| < r\}$. The technique of the next proof is adapted from that appearing in Theorem 10.30 of Rudin's text [21].

THEOREM 4.4.2. – *Let ω be a solitary-wave solution of* (4.0.1) *satisfying the conditions of* (4.4.1). *Then the corresponding velocity potential ϕ and the stream function ψ defined on $\overline{\Omega}$ both have analytic extensions as holomorphic functions on the open set*

$$\Omega_0 = \{(z_1, z_2) \in \mathbb{C}^2 : -\infty < x = \Re z_1 < \infty, |\Im z_1| < a, |\Re z_2| < H(x) + a, |\Im z_2| < a\}$$

for some constant $a > 0$.

Proof. – Using the fact verified in Theorem 4.3.4 that the second-order derivatives of X and Y are bounded on S_μ, it is seen that corresponding to $\epsilon = 1/2c$, there is a δ with $0 < \delta < \mu$ such that for any $(\phi, \psi) \in \overline{D_1}$, if (w_1, w_2), $(w_{10}, w_{20}) \in B((\phi, \psi), \delta)$, then

$$\left|\begin{pmatrix} X(w_1, w_2) - X(w_{10}, w_{20}) \\ Y(w_1, w_2) - Y(w_{10}, w_{20}) \end{pmatrix} - J_{(\phi,\psi)}\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right| \le \frac{1}{2c}\left|\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right|, \tag{4.4.3}$$

where

$$J_{(\phi,\psi)} = \begin{pmatrix} X_{w_1} & X_{w_2} \\ Y_{w_1} & Y_{w_2} \end{pmatrix}\Bigg|_{(\phi,\psi)},$$

c is the steady speed of the flow at infinity and δ is independent of the choice of $(\phi, \psi) \in \overline{D_1}$. Since

$$\left|J_{(\phi,\psi)}\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right| = \frac{1}{cq(\phi, \psi)}\left|\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right| > \frac{1}{c}\left|\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right|,$$

it follows from (4.4.3) and the Schwarz inequality that

$$\begin{aligned} &\left|\begin{pmatrix} X(w_1, w_2) - X(w_{10}, w_{20}) \\ Y(w_1, w_2) - Y(w_{10}, w_{20}) \end{pmatrix}\right| \\ &\qquad \ge \frac{1}{c}\left|\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right| - \frac{1}{2c}\left|\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right| = \frac{1}{2c}\left|\begin{pmatrix} w_1 - w_{10} \\ w_2 - w_{20} \end{pmatrix}\right|. \end{aligned} \tag{4.4.4}$$

Thus by the Open-Mapping Theorem (cf. [18]), $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a one-to-one mapping of the ball $B((\phi, \psi), \delta)$ onto an open set V, whose inverse is also a holomorphic mapping denoted by $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$. Let $x = X(\phi, \psi)$, and $y = Y(\phi, \psi)$. We shall show that there is an $a > 0$ independent of (ϕ, ψ) for which $B((x, y), a) \subset V$.

Let $(b_1, b_2) \in B((\phi, \psi), \delta)$ be any point such that $|(b_1, b_2) - (\phi, \psi)| < \delta/2$. Then $B((b_1, b_2), \delta/2) \subset B((\phi, \psi), \delta)$ and for any $(w_1, w_2) \in \partial B((b_1, b_2), \delta/2)$, it follows from (4.4.4) that

$$\left|\begin{pmatrix} X(w_1, w_2) - X(b_1, b_2) \\ Y(w_1, w_2) - Y(b_1, b_2) \end{pmatrix}\right| \ge \delta/4c. \tag{4.4.5}$$

Let $(\lambda_1, \lambda_2) \in B((X(b_1,b_2), Y(b_1,b_2)), \delta/8c)$. Then (4.4.5) and the choice of (λ_1, λ_2) yield the estimate

$$\min_{(w_1,w_2)\in\partial B((b_1,b_2),\delta/2)} \left| \begin{pmatrix} X(w_1,w_2)-\lambda_1 \\ Y(w_1,w_2)-\lambda_2 \end{pmatrix} \right| \geq \frac{\delta}{8c}.$$

On the other hand,

$$\left| \begin{pmatrix} X(b_1,b_2)-\lambda_1 \\ Y(b_1,b_2)-\lambda_2 \end{pmatrix} \right| < \frac{\delta}{8c}.$$

The last two facts and the Minimum-Value Principle imply that the mapping $\begin{pmatrix} X-\lambda_1 \\ Y-\lambda_2 \end{pmatrix}$ has a zero in the ball $B((b_1,b_2),\delta/2)$, which is to say $(\lambda_1,\lambda_2)\in V$. Since (λ_1,λ_2) is an arbitrary point in the ball $B((X(b_1,b_2),Y(b_1,b_2)),\delta/8c)$, it follows immediately that $B((X(b_1,b_2),Y(b_1,b_2)),\delta/8c) \subset V$. Thus it is concluded that $B((x,y),\delta/8c)\subset V$.

Hence, for any point (x,y), the velocity potential ϕ and the stream function ψ both have analytic extensions to holomorphic functions Φ and Ψ defined on the ball $B((x,y),\delta/8c)$. It follows from uniqueness of analytic extensions that ϕ and ψ can be extended as analytic functions to the set $\bigcup_{(x,y)\in\overline{D_1}} B((x,y),\delta/8c)$. Therefore, if the constant a is chosen as $a=\delta/16c$, say, then $\Omega_0 \subset \bigcup_{(x,y)\in\overline{D_1}} B((x,y),\delta/8c)$, which leads to the desired conclusion. □

Remark. – The analytic extensions Φ, Ψ of ϕ and ψ also satisfy the Cauchy-Riemann equations, a fact which may be verified as follows. At any point $(z_{10},z_{20})\in\Omega_0$, it follows from Theorem 4.4.2 that there is a neighbourhood V of (z_{10},z_{20}) and a neighbourhood U of $(w_{10},w_{20}) = (\Phi(z_{10},z_{20}),\Psi(z_{10},z_{20}))$ such that $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$ is a conformal mapping of V onto U. Differentiating the equations

$$\begin{aligned} z_1 &= X(w_1,w_2), \\ z_2 &= Y(w_1,w_2) \end{aligned}$$

with respect to z_1 and z_2, respectively, on V, where $w_1=\Phi(z_1,z_2)$ and $w_2=\Psi(z_1,z_2)$, leads to the relations

$$\begin{pmatrix} X_{w_1} & X_{w_2} \\ Y_{w_1} & Y_{w_2} \end{pmatrix}\begin{pmatrix} \Phi_{z_1} \\ \Psi z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} X_{w_1} & X_{w_2} \\ Y_{w_1} & Y_{w_2} \end{pmatrix}\begin{pmatrix} \Phi_{z_2} \\ \Psi z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Solving these equations shows that $\Phi_{z_1} = c^2q^2Y_{w_2}$, $\Phi_{z_2} = -c^2q^2X_{w_2}$, $\Psi_{z_1} = -c^2q^2Y_{w_1}$ and $\Psi_{z_2} = c^2q^2X_{w_1}$. The result now follows since X and Y satisfy the Cauchy-Riemann equations (see Theorem 4.3.3).

An immediate consequence of the last Remark is that the mapping $f=\phi+i\psi$ can be extended to a holomorphic function of one complex variable on the domain $\Omega_a = \{(x,y) : -\infty<x<\infty, -a-H(x)<y<H(x)+a\}$ in the complex plane.

COROLLARY 4.4.3. – *Under the conditions of Theorem* 4.4.2, *the conformal mapping* $w = f(z) = \phi(z) + i\psi(z)$ *defined for* z *in the flow domain* $\overline{\Omega}$ *has an analytic extension to the open set* $\Omega_a = \{z = x + iy : -\infty < x < \infty, -a - H(x) < y < H(x) + a\}$.

Proof. – The restriction $F(x, y)$ of the holomorphic function $F(z_1, z_2) = \Phi(z_1, z_2) + i\Psi(z_1, z_2)$ to the open set Ω_a has its real part $\Phi(x, y)$ and its imaginary part $\Psi(x, y)$ satisfying Cauchy-Riemann equations as verified in the preceding Remark. It follows that the restriction of $F(z_1, z_2)$ to Ω_a, as an extension of $f(z)$, is an analytic function of $z = x + iy$ on Ω_a. □

The last part of this section is focussed on analytic extensions of the functions which are determined by the streamlines in the domain $\overline{D_1}$.

THEOREM 4.4.4. – *For any number* $\lambda \in [-1, 1]$, *there exists a constant* $\rho > 0$ *such that the streamline* $\Psi(x, y) = \lambda$ *for* $(x, y) \in \overline{D_1}$ *defines a function* $y = H_\lambda(x)$ *which has an analytic extension to the strip* $\{z \in \mathbb{C} : |\Im z| < \rho\}$. *In particular, when* $\lambda = 1$, *this extension is that previously obtained for the solitary-wave profile* $H = H_1$.

Proof. – It follows from Theorem 4.4.1 and Theorem 4.4.2 that the holomorphic mapping $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a homeomorphism of the line $\{-\infty < \phi < \infty, \psi = \lambda\}$ onto the streamline $\psi(x, y) = \lambda$, and at each point (x_0, y_0) on the stream line, ψ has an analytic extension as a holomorphic function $\Psi(z_1, z_2)$ of two complex variables in a neighbourhood of (x_0, y_0). Since $\psi_y = \phi_x = cq\cos\omega \neq 0$ on the domain $\overline{D_1}$, applying the Implicit-Function Theorem leads to the conclusion that there is a neighbourhood $V_0 \subset \mathbb{C}$ of x_0 on which there is a unique holomorphic function $H_\lambda(z)$ satisfying the equation $\Psi(z, H_\lambda(z)) = \lambda$ and the condition $H_\lambda(x_0) = y_0$. Here, we want to show that there is a constant $\rho > 0$ independent of the choice of (x_0, y_0) on the streamline, such that the disk $D(x_0, \rho) = \{z \in \mathbb{C} : |z - x_0| < \rho\} \subset V_0$. This fact implies the stated conclusion.

The latter point may be proved by using Theorem 4.3.4 and Theorem 4.4.2. Notice that at each point (x_0, y_0) on the stream line, $H_\lambda(z)$ satisfies the equation

$$H_\lambda(z) = y_0 + \int_{x_0}^{z} \tan\theta(\Phi(t, H_\lambda(t)), \Psi(t, H_\lambda(t)))dt$$

on V_0, where $\theta = -\omega$. When using a fixed-point theorem to show existence and uniqueness of the solution H_λ in a disk $D(x_0, \rho)$ about x_0, if the radius ρ depends only on bounds for the function $\tan\theta(\Phi, \Psi)$, the first order partial derivatives of $\tan\theta(\Phi, \Psi)$ and the neighbourhood of (x_0, y_0) where the analytic extensions Φ and Ψ of ϕ and ψ are defined, then the desired result follows. It was proved in Theorem 4.4.2 that Φ and Ψ are defined in a neighbourhood of (x_0, y_0) that contains the ball $B((x_0, y_0), \delta/8c)$ with δ independent of the choice of $(x_0, y_0) \in \overline{D_1}$, and $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$ maps $B((x_0, y_0), \delta/8c)$ into the ball $B((\phi_0, \lambda), \delta)$ contained in S_μ, where the functions mentioned above are all bounded as proved in Theorem 4.3.4. Therefore, the choice of ρ can also be made independently of (x_0, y_0) and the theorem is established. □

5. Conclusion

Two aspects of solitary-wave solutions of nonlinear evolution equations have been considered here, namely the way they decay to a quiescent state away from their crest and their regularity. These properties have been studied by viewing the relevant solitary wave as a solution of a nonlinear convolution equation.

The regularity results show that solitary-wave solutions of nonlinear evolution equations are generally real-analytic functions that are the restriction to the real axis of functions holomorphic in a strip in the complex plane centered about the real axis. The theory obtained here broadens considerably the range of convolution equations for which this conclusion is valid. One consequence is that the dependent variables corresponding to solitary-wave solutions of the two-dimensional Euler equations for gravity waves in a channel are shown to be the restriction of analytic functions.

The decay of solitary-wave solutions is seen to depend very strongly on the dispersion relation appearing in the particular evolution equation. Our theory for this aspect appears to be sharp in its application, at least in some cases. Again, in addition to model equations, the two-dimensional Euler equations for surface water waves fall within the purview of our general results.

Because the solitary-solutions of the Euler equations are the restriction of analytic functions, it might be hoped that the same qualitative feature will be inherited by their good approximations, which is to say solitary-wave solutions of approximate evolution equations. Indeed, model equations like the KdV-equation are derived from the Euler equations by making a formal Taylor expansion of the solution with regard to certain naturally arising small parameters (cf. [23]). The coefficients of this expansion are functions of the physical variables, and since their sum is an analytic function of these variables, it would be surprising if the individual coefficients were not likewise analytic. Our theory shows that for solitary-wave solutions, at least the lowest-order coefficients do indeed possess this property. It would be interesting to extend the work of Friedrichs and Hyers [8] showing that Euler-equation solitary waves are, for Froude numbers close to one, well approximated by the KdV-equation solitary waves. We have in mind a result showing that these two solitary-wave solutions, when viewed as analytic functions on a strip in $\mathbb{C}$, become close as F nears 1.

Appendix

Derivatives of composite functions

Proof of Lemma 4.1.1. – Induction will be used to verify the following identity, which is valid for any integer $n \geq 2$:

$$\text{(A1)} \quad \frac{d^n f(g(x))}{dx^n} = y^{(n)} f'(y) + \sum_{s=2}^{n} \frac{f^{(s)}(y)}{s!} \sum^{(s,n)} \binom{n}{j_1, \cdots, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots y^{(j_{s-1})},$$

where $y = g(x)$, $y^{(k)} = g^{(k)}(x)$ and

$$\sum^{(s,n)} = \sum_{j_1=s-1}^{n-1} \sum_{j_2=s-2}^{j_1-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}-1} .$$

When $n = 2, 3$, the right-hand sides of (A1) are

$$y'' f'(y) + (y')^2 f''(y)$$

and

$$\begin{aligned} &y^{(3)} f'(y) + \frac{f''(y)}{2} \sum_{j=1}^{2} \binom{3}{j} y^{(3-j)} y^{(j)} + (y')^3 f^{(3)}(y) \\ &= y^{(3)} f'(y) + (y')^3 f^{(3)}(y) + 3y'' y' f''(y), \end{aligned}$$

respectively. Thus (A1) holds for $n = 2, 3$ by inspection. Suppose that there is an integer $n \geq 3$ such that Equality (A1) holds. Taking the derivative with respect to x of both sides of Equality (A1), we have:

$$\begin{aligned} \text{(A2)} \qquad \frac{d^{n+1} f(g(x))}{dx^{n+1}} &= y^{(n+1)} f'(y) + y^{(n)} y' f''(y) \\ &+ \sum_{s=2}^{n} \frac{f^{(s+1)}(y)}{s!} y' \sum^{(s,n)} \binom{n}{j_1, \cdots, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots y^{(j_{s-1})} \\ &+ \sum_{s=2}^{n} \frac{f^{(s)}(y)}{s!} \sum^{(s,n)} \binom{n}{j_1, \cdots, j_{s-1}} \left(y^{(n-j_1)} y^{(j_1-j_2)} \cdots y^{(j_{s-1})} \right)'. \end{aligned}$$

To simplify the right-hand side of (A2), first consider the following expression for an s with $2 \leq s \leq n$.

$$\begin{aligned} \text{(A3)} \qquad &\sum^{(s,n)} \binom{n}{j_1, \cdots, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots (y^{(j_{s-2}-j_{s-1})} y^{(j_{s-1})})' \\ &= \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{n}{j_1, \cdots, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots \\ &\quad \cdots y^{(j_{s-2}-j_{s-1})} y^{(j_{s-1}+1)} \\ &\quad + \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{n}{j_1, \cdots, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots \\ &\quad \cdots y^{(j_{s-2}+1-j_{s-1})} y^{(j_{s-1})} \\ &= \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=2}^{j_{s-2}} \binom{n}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}-1} y^{(n-j_1)} y^{(j_1-j_2)} \cdots \end{aligned}$$

$$\cdots y^{(j_{s-2}+1-j_{s-1})} y^{(j_{s-1})}$$

$$+ \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots$$

$$\cdots y^{(j_{s-2}+1-j_{s-1})} y^{(j_{s-1})}$$

$$= \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \sum_{j_{s-1}=1}^{j_{s-2}} \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{s-2}+1}{j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots$$

$$\cdots y^{(j_{s-2}+1-j_{s-1})} y^{(j_{s-1})} +$$

$$-2 \sum_{j_1=s-1}^{n-1} \sum_{j_2=s-2}^{j_1-1} \cdots \sum_{j_{s-2}=2}^{j_{s-3}-1} \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{s-3}}{j_{s-2}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots$$

$$\cdots y^{(j_{s-3}-j_{s-2})} y^{(j_{s-2})} y'.$$

When $s = 2$, formula (A3) yields

$$\text{(A4)} \qquad \sum_{j=1}^{n-1} \binom{n}{j} \left(y^{(n-j)} y^{(j)} \right)' = \sum_{j=1}^{n} \binom{n+1}{j} y^{(n+1-j)} y^{(j)} - 2y' y^{(n)}.$$

Therefore, the following identity holds for $l = s - 2$:

$$\text{(A5)} \qquad \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_l=s-l}^{j_{l-1}-1} \sum_{j_{l+1}=s-l-1}^{j_l-1} \cdots \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{n}{j_1, \ldots, j_{s-1}} y^{(n-j_1)} \cdots$$

$$\cdots y^{(j_{l-1}-j_l)} \left(y^{(j_l-j_{l+1})} \cdots y^{(j_{s-2}-j_{s-1})} y^{(j_{s-1})} \right)'$$

$$= \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_l=s-l}^{j_{l-1}-1} \sum_{j_{l+1}=s-l-1}^{j_l} \cdots \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{n}{j_1} \cdots$$

$$\cdots \binom{j_{l-1}}{j_l} \binom{j_l+1}{j_{l+1}} \binom{j_{l+1}}{j_{l+2}} \cdots \binom{j_{s-2}}{j_{s-1}} y^{(n-j_1)} \cdots$$

$$\cdots y^{(j_{l-1}-j_l)} y^{(j_l+1-j_{l+1})} y^{(j_{l+1}-j_{l+2})} \cdots y^{(j_{s-1})} +$$

$$-(s-l) \sum_{j_1=s-1}^{n-1} \cdots \sum_{j_l=s-l}^{j_{l-1}-1} \sum_{j_{l+1}=s-l-2}^{j_l-1} \cdots \sum_{j_{s-2}=1}^{j_{s-3}-1} \binom{n}{j_1, \ldots, j_{s-2}} \cdot$$

$$\cdot\, y^{(n-j_1)} y^{(j_1-j_2)} \cdots y^{(j_{s-3}-j_{s-2})} y^{(j_{s-2})} y'.$$

Suppose that there is an integer l with $1 < l \le s - 2$ such that (A5) holds. Then use (A5) to simplify the following expression:

$$\sum_{j_1=s-1}^{n-1} \cdots \sum_{j_{l-1}=s-l+1}^{j_{l-2}-1} \cdots \sum_{j_{s-1}=1}^{j_{s-2}-1} \binom{n}{j_1, \ldots, j_l, \ldots, j_{s-1}} y^{(n-j_1)} y^{(j_1-j_2)} \cdots$$

$$\cdots\left(y^{(j_{l-1}-j_l)}y^{(j_l-j_{l+1})}\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_{s-1})}\right)'$$

$$= \sum_{j_1=s-1}^{n-1}\cdots\sum_{j_l=s-l}^{j_{l-1}-1}\cdots\sum_{j_{s-1}=1}^{j_{s-2}-1}\binom{n}{j_1,\cdots,j_l,\cdots,j_{s-1}}y^{(n-j_1)}y^{(j_1-j_2)}\cdots$$
$$\cdots y^{(j_{l-1}+1-j_l)}y^{(j_l-j_{l+1})}\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_{s-1})}$$
$$+\sum_{j_1=s-1}^{n-1}\cdots\sum_{j_{l-1}=s-l+1}^{j_{l-2}-1}\sum_{j_l=s-l+1}^{j_{l-1}}\sum_{j_{l+1}=s-l-1}^{j_l-1}\cdots\sum_{j_{s-1}=1}^{j_{s-2}-1}\binom{n}{j_1}\cdots$$
$$\cdots\binom{j_{l-1}}{j_l-1}\binom{j_l}{j_{l+1}}\binom{j_{l+1}}{j_{l+2}}\cdots\binom{j_{s-2}}{j_{s-1}}y^{(n-j_1)}\cdots$$
$$\cdots y^{(j_{l-1}+1-j_l)}y^{(j_l-j_{l+1})}\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_{s-1})}+$$
$$-(s-l)\sum_{j_1=s-1}^{n-1}\cdots\sum_{j_l=s-l}^{j_{l-1}-1}\sum_{j_{l+1}=s-l-2}^{j_l-1}\cdots\sum_{j_{s-2}=1}^{j_{s-3}-1}\binom{n}{j_1,\cdots,j_{s-2}}\cdot$$
$$\cdot y^{(n-j_1)}y^{(j_1-j_2)}\cdots y^{(j_{s-3}-j_{s-2})}y^{(j_{s-2})}y'$$
$$=\sum_{j_1=s-1}^{n-1}\cdots\sum_{j_{l-1}=s-l+1}^{j_{l-2}-1}\sum_{j_l=s-l}^{j_{l-1}}\sum_{j_{l+1}=s-l-1}^{j_l-1}\cdots\sum_{j_{s-1}=1}^{j_{s-2}-1}\binom{n}{j_1}\cdots$$
$$\cdots\binom{j_{l-1}+1}{j_l}\binom{j_l}{j_{l+1}}\binom{j_{l+1}}{j_{l+2}}\cdots\binom{j_{s-2}}{j_{s-1}}y^{(n-j_1)}\cdots$$
$$\cdots y^{(j_{l-1}+1-j_l)}y^{(j_l-j_{l+1})}y^{(j_{l+1}-j_{l+2})}\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_{s-1})}$$
$$-(s-l+1)\sum_{j_1=s-1}^{n-1}\cdots\sum_{j_l=s-l-1}^{j_{l-1}-1}\sum_{j_{l+1}=s-l-2}^{j_l-1}\cdots\sum_{j_{s-2}=1}^{j_{s-3}-1}\binom{n}{j_1,\cdots,j_{s-2}}\cdot$$
$$\cdot y^{(n-j_1)}y^{(j_1-j_2)}\cdots y^{(j_{s-3}-j_{s-2})}y^{(j_{s-2})}y'.$$

Continuing this procedure for descending integers l from $l=s-2$ to $l=1$ allows one to conclude

$$\text{(A6)}\qquad \sum\nolimits^{(s,n)}\binom{n}{j_1,\cdots,j_{s-1}}(y^{(n-j_1)}\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_s)})'$$
$$=\sum_{j_1=s-1}^{n}\cdots\sum_{j_{s-1}=1}^{j_{s-2}-1}\binom{n+1}{j_1,\cdots,j_{s-1}}y^{(n+1-j_1)}y^{(j_1-j_2)}\cdots$$
$$\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_{s-1})}+$$
$$-s\sum_{j_1=s-2}^{n-1}\sum_{j_2=s-3}^{j_1-1}\cdots\sum_{j_{s-2}=1}^{j_{s-3}-1}\binom{n}{j_1,\cdots,j_{s-2}}y^{(n-j_1)}y^{(j_1-j_2)}\cdots$$
$$\cdots y^{(j_{s-3}-j_{s-2})}y^{(j_{s-2})}y'$$
$$=\sum\nolimits^{(s,n+1)}\binom{n+1}{j_1,\cdots,j_{s-1}}y^{(n+1-j_1)}\cdots y^{(j_{s-2}-j_{s-1})}y^{(j_{s-1})}+$$

$$- s \sum^{(s-1,n)} \binom{n}{j_1, \cdots, j_{s-2}} y^{(n-j_1)} \cdots y^{(j_{s-3}-j_{s-2})} y^{(j_{s-2})} y'.$$

Substituting (A4) and (A6) into (A2) leads to

$$\begin{aligned}
\frac{d^{n+1} f(g(x))}{dx^{n+1}} &= y^{(n+1)} f'(y) + y^{(n)} y' f''(y) + \\
&+ \sum_{s=3}^{n+1} \frac{f^{(s)}(y) y'}{(s-1)!} \sum^{(s-1,n)} \binom{n}{j_1, \cdots, j_{s-2}} y^{(n-j_1)} \cdots y^{(j_{s-2})} + \\
&+ \sum_{s=2}^{n} \frac{f^{(s)}(y)}{s!} \Bigg(\sum^{(s,n+1)} \binom{n+1}{j_1, \cdots, j_{s-1}} y^{(n+1-j_1)} \cdots y^{(j_{s-1})} + \\
&- s \sum^{(s-1,n)} \binom{n}{j_1, \cdots, j_{s-2}} y^{(n-j_1)} \cdots y^{(j_{s-2})} y' \Bigg) \\
&= y^{(n+1)} f'(y) + \sum_{s=2}^{n+1} \frac{f^{(s)}(y)}{s!} \sum^{(s,n+1)} \binom{n+1}{j_1, \cdots, j_{s-1}} y^{(n+1-j_1)} \cdots y^{(j_{s-1})}.
\end{aligned}$$

Therefore, (A1) holds for any integer $n \geq 2$ by induction. □

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